Monetary Policy and Rational Asset Price Bubbles: Comments

By Jianjun Miao, Zhouxiang Shen, and Pengfei Wang*

We revisit Galí's (2014) analysis by extending his model to incorporate persistent bubble shocks. We find that, under adaptive learning, a stable bubbly steady state and the associated sunspot solutions under optimal monetary policy are not E-stable. When deriving the unique forward-looking minimum stable variable (MSV) solution around an unstable bubbly steady state, we obtain results that are consistent with the conventional views: leaning-against-the-wind policy reduces bubble volatility and is optimal. Such a steady state and the associated MSV solution are E-stable.

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Due to the recent financial crisis during 2008-2009, there is a renewed interest in understanding the role of asset bubbles in business cycles and the associated policy implications. Galí (2014) presents an elegant overlapping generations model with nominal rigidities to study the impact of monetary policy on rational asset bubbles. He finds some intriguing results that are inconsistent with conventional views. These results are summarized below:

- A stronger interest rate response to bubble fluctuations (i.e., a "leaning against the wind policy") may raise the volatility of asset prices and of their bubble component.
- The optimal monetary policy strikes a balance between stabilization of current aggregate demand and the bubble. If the average size of the bubble is sufficiently large, the latter motive will be dominant, making it optimal for the central bank to lower interest rates in the face of a growing bubble.

In this paper we revisit Galí's analysis by extending his model to allow for serially correlated bubble shocks. Our analysis complements his. We argue that his results are driven by his particular choice of the equilibrium solution. In his

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model there are multiple steady states and multiple equilibria. In particular, there is a continuum of stable bubbly steady states and a continuum of unstable bubbly steady states. He focuses on a backward-looking sunspot solution around a stable bubbly steady state. For this solution the value of a pre-existing asset bubble only responds to its own innovations. In the absence of such innovations, the size of an old bubble is predetermined, and an increase in the interest rate will raise its future size. By contrast we analyze the forward-looking minimal state variable (MSV) solution around an unstable bubbly steady state. For this solution the asset bubble responds to shocks on impact just like any asset prices. An increase in interest rates dampens the asset bubble on impact. We find results that are consistent with conventional views and are different from Galí's results mentioned above. In particular, the optimal policy calls for a leaning-against-the-wind rule. Note that this result depends on the assumption of serially correlated bubble shocks. If bubble shocks are serially uncorrelated, monetary policy would not affect bubble volatility for the MSV solution.

All steady states and equilibria in Galí's model are consistent with rational expectations. Following the methodology surveyed by Evans and Honkapohja (1999, 2001), we use learning as a selection device to select a particular steady state and a particular equilibrium.[1] The idea is that agents of the model do not initially have rational expectations and they instead form forecasts by using some adaptive learning rules such as recursive least squares based on the data. The question is whether the agents can learn a particular equilibrium or a particular steady state. Marcet and Sargent (1989) and Evans and Honkapohja (1999, 2001) show that the notion of expectational stability (E-stability) determines local convergence of real time recursive learning algorithms in a wide variety of economic models.

We find that the sunspot equilibrium solution adopted by Galí (2014) is not E-stable under his optimal monetary policy rule, but the forward-looking MSV solution is E-stable. We also find that the unstable bubbly steady state Pareto dominates the stable bubbly steady state. Moreover the former steady state is E-stable, but the latter is not. Our results are analogous to those in Evans, Honkapohja, and Marimon (2007, 2001). They show that the E-unstable high-inflation steady state in a hyperinflation model has counterintuitive policy implications, while the E-stable low inflation steady state has conventional implications.

I. Solving Galí's Model

We first summarize Galí's (2014) model and refer the reader to his paper for detailed economic interpretations. We extend his model by allowing for persistent bubble shocks. We then solve for all equilibria and select equilibrium using a learning device.

[1]See Bullard and Mitra (2002), Adam (2003), Woodford (2003), Duffy and Xiao (2007), Benhabib, Evans, and Honkapohja (2012), Christiano, Eichenbaum, and Johannsen (2018), among others, for the application of learning to select equilibrium in macroeconomic models.

A. Setup

The model economy consists of overlapping generations of agents, firms, and a central bank. Each agent lives for two periods and an agent born in period t derives utility according to $\log C_{1,t}+\beta E_t\left[\log C_{2,t+1}\right]$, where $C_{1,t}=\left(\int_0^1 C_{1,t}\left(i\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}$ and $C_{2,t+1}=\left(\int_0^1 C_{2,t+1}\left(i\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}$ are consumption bundles and $\epsilon>1$. Each young agent is endowed with one unit of labor and supplies it to firms inelastically. Normalize the size of each cohort to unity.

Each young agent is endowed with $\delta\in(0,1)$ units of an intrinsically useless bubble asset. The bubble asset can be traded in an asset market. Each period a fraction δ of each vintage of bubble assets loses its value so that the total amount of bubble assets outstanding remains constant and equal to one. This modeling allows a new bubble to be created once an old bubble bursts, as in Martin and Ventura (2012), Wang and Wen (2012), and Miao, Wang, and Xu (2015).

An agent born in period t chooses differentiated consumption goods $C_{1,t}\left(i\right)$ and $C_{2,t+1}\left(i\right)$, bond holdings Z_t^M, and holdings $Z_{t|t-k}^B$ of bubble asset introduced in period $t-k$ to maximize utility subject to the following budget constraints

$$\int_0^1\frac{P_t\left(i\right)C_{1,t}\left(i\right)}{P_t}di+\frac{Z_t^M}{P_t}+\sum_{k=0}^{\infty}Q_{t|t-k}^BZ_{t|t-k}^B=W_t+\delta Q_{t|t}^B, \tag{1}$$

$$\int_0^1\frac{P_{t+1}\left(i\right)C_{2,t+1}\left(i\right)}{P_{t+1}}di=D_{t+1}+\frac{Z_t^M\left(1+i_t\right)}{P_{t+1}}+\left(1-\delta\right)\sum_{k=0}^{\infty}Q_{t+1|t-k}^BZ_{t|t-k}^B,$$

where $P_t=\left(\int_0^1 P_t\left(i\right)^{1-\epsilon}di\right)^{\frac{1}{1-\epsilon}}$ is the consumption price index, W_t is the real wage, i_t is the nominal interest rate, D_{t+1} is firm dividends, and $Q_{t|t-k}^B$ is the period-t real price of the bubble asset introduced in period $t-k$. Define the gross real interest rate as

$$R_t=\left(1+i_t\right)E_t\frac{1}{\Pi_{t+1}}. \tag{2}$$

Each agent owns a firm that produces a differentiated product $Y_t\left(i\right)$ using labor input $N_t\left(i\right)$ according to the technology $Y_t\left(i\right)=N_t\left(i\right)$. Each firm is monopolistically competitive and sets price P_t^* one period in a advance, generating nominal rigidities. It solves the following problem

$$\max_{P_t^*}E_{t-1}\left[\frac{\beta C_{1,t-1}}{C_{2,t}}Y_t\left(i\right)\left(\frac{P_t^*}{P_t}-W_t\right)\right]$$

subject to the demand schedule $Y_t\left(i\right)=\left(P_t^*/P_t\right)^{-\epsilon}C_t$, where $C_t=C_{1,t}+C_{2,t}$. In a symmetric equilibrium we have

$$0 = E_{t-1}\left[\frac{\beta C_{1,t-1}}{C_{2,t}}\left(1-\mathcal{M}W_t\right)\right], \tag{3}$$

where $\mathcal{M} = \epsilon/\left(\epsilon-1\right)$ denotes the markup.

The labor and goods markets clearing implies

$$C_{1,t} + C_{2,t} = 1, \tag{4}$$
$$D_t + W_t = 1. \tag{5}$$

Asset market clearing requires $Z_t^M = 0$ and $Z_{t|t-k}^B = \delta\left(1-\delta\right)^k$. Define the aggregate bubble index Q_t and the old bubble index B_t as

$$Q_t = \delta\sum_{k=0}^{\infty}\left(1-\delta\right)^k Q_{t|t-k}^B,\ \ B_t = \delta\sum_{k=1}^{\infty}\left(1-\delta\right)^k Q_{t|t-k}^B.$$

Let $U_t = \delta Q_{t|t}^B$ denote the size of new bubbles. Then by definition and the agent's bubble asset choice condition,

$$Q_t = B_t + U_t, \tag{6}$$
$$B_t + U_t = \beta E_t\left[\frac{C_{1,t}}{C_{2,t+1}}B_{t+1}\right]. \tag{7}$$

The consumption Euler equation gives

$$1 = \beta\left(1+i_t\right)E_t\left[\frac{C_{1,t}}{C_{2,t+1}}\frac{1}{\Pi_{t+1}}\right]. \tag{8}$$

The budget constraint (1) and the market-clearing conditions imply

$$C_{1,t} + Q_t = W_t + U_t. \tag{9}$$

To close the model, the central bank sets the nominal interest rate according to a feedback rule, which may respond to asset bubbles,

$$\ln\left(1+i_t\right) = \ln R + \phi_\pi \ln\left(\frac{\Pi_t}{\Pi}\right) + \phi_b \ln\left(\frac{Q_t}{Q}\right) + \ln E_t\Pi_{t+1}, \tag{10}$$

where $\phi_\pi > 0$, $\Pi_t = P_t/P_{t-1}$ denotes gross inflation, and a variable without time subscript denotes its steady-state value. The central questions are how monetary policy affects asset bubbles and whether monetary policy should respond to asset bubbles.

The equilibrium system consists of eight equations (3), (4), (5), (6), (7), (8), (9), and (10) for nine stochastic processes $\{C_{1,t}\}, \{C_{2,t}\}, \{D_t\}, \{W_t\}, \{\Pi_t\}, \{i_t\}, \{Q_t\}, \{B_t\}$, and $\{U_t\}$. Since there are eight equilibrium conditions for nine variables, the equilibrium system cannot determine the size of the old bubble and the new bubble independently. Galí (2014) assumes that the new bubble $\{U_t\}$ is an exogenously given IID process. We consider the more general case in which $\{U_t\}$ is serially correlated. Galí (2014) also considers the innovation in the old bubble $B_t - E_{t-1}B_t$ as another independent source of uncertainty. We will show below that this is true for the sunspot equilibria. Except for these two sources of uncertainty, there is no other shock in the model.

B. *Multiple Equilibria*

We first present Galí's results in the deterministic case where $U_t = U > 0$ for all t. Then the old bubble $\{B_t\}$ satisfies the difference equation

$$B_{t+1} = \frac{(1 - 1/\mathcal{M})(B_t + U)}{\beta/\mathcal{M} - (1+\beta) B_t - U} \equiv H(B_t, U). \tag{11}$$

The necessary and sufficient condition for the existence of a deterministic bubbly steady state is given by

$$\mathcal{M} < 1 + \beta. \tag{12}$$

Furthermore, when this condition is satisfied there exists a continuum of stable bubbly steady states indexed by U,

$$\left\{(B_s(U), U) : B_s(U) = H(B_s(U), U) \text{ for } U \in \left(0, \bar{U}\right)\right\},$$

and a continuum of unstable bubbly steady states also indexed by U,

$$\left\{(B_u(U), U) : B_u(U) = H(B_u(U), U) \text{ for } U \in [0, \bar{U}),\ B_u(U) > B_s(U)\right\},$$

where

$$\bar{U} = \beta + (1+\beta)(1-W) - 2\sqrt{\beta(1+\beta)(1-W)} > 0 \text{ and } W = \frac{1}{\mathcal{M}}.$$

The economy also has a bubbleless steady state in which $B = U = 0$. In this steady state we can show the bubbleless real interest rate is $R_f = (\mathcal{M} - 1)/\beta$. Thus condition (12) is the same as $R_f < 1$, which is the standard condition in the literature (Tirole (1985)), i.e., the bubbleless equilibrium is dynamically inefficient.

Next we study the stochastic case by log-linearizing the equilibrium system around a deterministic bubbly steady state for a fixed $U \in \left(0, \bar{U}\right)$. In Appendix A we show that the log-linearized equilibrium system can be reduced to a unidimensional system

$$b_t = \frac{1}{R(\phi_b+1)}E_t b_{t+1} + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}E_{t-1}b_t + \frac{R-1}{R}u_t + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}E_{t-1}u_t, \tag{13}$$

where we use a lower case variable to denote the log deviation from its steady-state value and R denotes the bubbly steady-state real interest rate given by

$$R = \frac{1}{\beta}\frac{1-1/\mathcal{M}+B}{1/\mathcal{M}-B} = \frac{B}{B+U} \in (0,1)\,.$$

Note that there are two bubbly steady states for a fixed $U \in \left(0,\bar{U}\right)$. Without risk of confusion, we use the same notation B to represent either one of the steady-state size of the old bubble in the analysis below.

Our objective is to solve for a rational expectations equilibrium (REE) using (13). Galí (2014) assumes that u_t is IID. We consider a more general AR(1) process

$$u_t = \rho u_{t-1} + e_t, \quad \rho \in [0,1), \tag{14}$$

where e_t is an IID random variable with mean zero and variance σ_e^2.

Galí (2014) focuses his analysis on a sunspot solution around a stable bubbly steady state. Given (14), we can derive the following more general solution. Its proof and the proofs of the remaining propositions in the paper are given in Appendix B.

PROPOSITION 1: *Fix $U \in \left(0,\bar{U}\right)$. For any b_0, there is a linear sunspot solution in a neighborhood of the bubbly stable steady state given by*

$$b_t = \chi b_{t-1} + (1-R)\left(1+\epsilon B\right)\rho u_{t-2} + \varphi_2^* e_t + \varphi_3^* e_{t-1} + \varphi_4^*\xi_t + \varphi_5^*\xi_{t-1},$$

where ξ_t denotes a sunspot shock satisfying $E_{t-1}\xi_t = 0$, φ_3^* and φ_5^* are arbitrary real numbers, and

$$\varphi_2^* = \frac{\varphi_3^* + (R-1)\left(1+\phi_b\right)}{R\left(\phi_b+1\right)-\chi},\ \varphi_4^* = \frac{\varphi_5^*}{R(\phi_b+1)-\chi},$$

$$\chi = R\left(1+\epsilon B\left(1+\beta\right)\right) \in (0,1).$$

Galí (2014) shows that $\chi = \partial H\left(B,U\right)/\partial B$. For a stable bubbly steady state, we must have $\chi \in (0,1)$, which also implies that the backward-looking solution in Proposition 1 is stationary. Galí (2014) defines a sunspot variable $\xi_t = b_t - E_{t-1}b_t$.

Substituting this variable into (13) yields a particular solution

$$\begin{aligned} b_t &= \chi b_{t-1} + (\phi_b + 1)(1-R)\,u_{t-1} - (\phi_b - \epsilon B)(1-R)\rho u_{t-2} \\ &\quad + \xi_t + (\phi_b - \epsilon B(1+\beta))\,R\xi_{t-1}, \end{aligned} \tag{15}$$

which can also be obtained by setting

$$\varphi_2^* = 0, \quad \varphi_3^* = (1-R)(1+\phi_b), \ \varphi_5^* = (\phi_b - \epsilon B(1+\beta))\,R$$

in our general solution given in Proposition 1. The solution in equation (30) of Galí (2014) corresponds to $\rho = 0$ in (15).

For this solution, the initial value b_0 is indeterminate. Galí (2014) derives all his results for a fixed b_0. From (15) we can see that monetary policy only affects the anticipated component of the old bubble $E_{t-1}b_t$ through the interest rate coefficient ϕ_b. In the case of $\rho = 0$, Galí (2014) shows that a leaning-against-the-wind policy which corresponds to $\phi_b > 0$ generates a larger volatility in the bubble than a policy of benign neglect ($\phi_b = 0$).

Now we consider the solution in the neighborhood of the unstable bubbly steady state.

PROPOSITION 2: *Fix $U \in \left(0, \bar{U}\right)$. There is a unique forward-looking linear solution in a neighborhood of the unstable bubbly steady state given by*

$$b_t = (R-1)\frac{\epsilon B + 1}{\chi - \rho}\rho u_{t-1} + \frac{R-1}{R}\left[\frac{\rho}{1+\phi_b}\frac{1+\epsilon B}{\chi - \rho} + 1\right] e_t, \tag{16}$$

where $\chi = R(1 + \epsilon B(1+\beta)) > 1$.

In a neighborhood of the unstable bubbly steady state, we have $\chi > 1$. The backward-looking solution in (15) is not stationary. We must solve for b_t forward to obtain the forward-looking solution in (16) so that b_t is stationary. This solution is also called the minimal state variable (MSV) solution in the literature (e.g., Evans and Honkapohja (2001)). In the next section we will focus our analysis on this solution.

Note that if $\rho = 0$ as in Galí (2014), then the MSV solution gives $b_t = e_t(R-1)/R$. In this case monetary policy through ϕ_b does not affect bubble dynamics. We thus assume $\rho \in (0,1)$ throughout the paper.

C. *Learning and Equilibrium Selection*

There are multiple (deterministic) steady states and multiple REE solutions in Galí (2014). We will use learning as a selection device to select a particular steady state and a particular REE solution. To understand the basic idea, we consider an economic model with a solution described as a particular parameter vector $\bar{\varphi}$ (e.g., the parameters of an autoregressive process or a steady state).

Under adaptive learning agents do not know $\bar{\varphi}$ but estimate it from data using a statistical procedure such as least squares. This leads to estimates φ_t at time t and the question is whether $\varphi_t \to \bar{\varphi}$ as $t \to \infty$. Evans and Honkapohja (2001) show that, for a wide range of economic examples and learning rules, convergence is governed by the corresponding E-stability condition, i.e., the local asymptotic stability of $\bar{\varphi}$ under the differential equation

$$\frac{d\varphi}{d\tau} = T(\varphi) - \varphi, \tag{17}$$

where τ denotes notional or virtual time, $T(\varphi)$ is the mapping from the perceived law of motion (PLM) φ to the implied actual law of motion (ALM) $T(\varphi)$. In the following analysis we will check the E-stability condition.

We start by the steady states.

PROPOSITION 3: *For any fixed $U \in \left(0, \bar{U}\right)$, the bubbly unstable steady state Pareto dominates the bubbly stable steady state. Moreover the bubbly unstable steady state is E-stable if and only if $\phi_b > -1$ and the bubbly stable steady state is E-stable if and only if $\phi_b < -1$.*[2]

Next we consider the stochastic MSV and sunspot solutions.

PROPOSITION 4: *For $\phi_b > -1$ the sunspot solution in Proposition 1 is not E-stable. The MSV solution in Proposition 2 is E-stable if and only if $\phi_b > -1$.*

Galí (2014) shows that the optimal response coefficient ϕ_b that minimizes the welfare loss is greater than -1 for the sunspot solution. Proposition 4 shows that this solution under the optimal policy is not E-stable. By contrast, the MSV solution for $\phi_b > -1$ is E-stable. In the next section we will show that the optimal coefficient ϕ_b is positive for the MSV solution and hence the MSV solution under optimal monetary policy is E-stable.

II. Monetary Policy

What is the impact of the monetary policy on bubble dynamics? We first use (16) to compute the volatility of the old bubble for the MSV solution

$$Var(b_t) = \left(\frac{\epsilon B + 1}{\chi - \rho}\rho\right)^2 \frac{(R-1)^2 \sigma_e^2}{1 - \rho^2} + \left(\frac{R-1}{R}\right)^2 \left[\frac{\rho}{1+\phi_b}\frac{1+\epsilon B}{\chi - \rho} + 1\right]^2 \sigma_e^2.$$

It is minimized at

$$\phi_b = -\frac{\rho(1+\epsilon B)}{\chi - \rho} - 1 < -1. \tag{18}$$

[2] In a previous version of the paper, we started with the deterministic system (11) directly. The PLM is $B_{t+1} = a$ and the ALM is $T(a) = H^{-1}(a, U)$. The ODE is $\dot{a} = T(a) - a$. In this case the assumption on ϕ_b is not needed.

Galí (2014) shows that the volatility of both the old and aggregate bubbles is minimized at $\phi_b = -1$ for his sunspot solution.

Now we log-linearize equation (6) to obtain

$$q_t = Rb_t + (1 - R)\, u_t \tag{19}$$

and combine it with (16) to derive the volatility of the aggregate bubble for the MSV solution:

$$Var\,(q_t) = (R-1)^2 \left[R \frac{\epsilon B + 1}{\chi - \rho} - 1 \right]^2 \rho^2 (1-\rho^2)^{-1} \sigma_e^2 + \left[\frac{(R-1)\,\rho}{1+\phi_b} \frac{1+\epsilon B}{\chi - \rho} \right]^2 \sigma_e^2.$$

Thus a leaning-against-the-wind policy (i.e., $\phi_b > 0$) generates a lower volatility of the aggregate bubble than a policy of benign neglect ($\phi_b = 0$), contrary to Galí's result. The volatility is minimized when $\phi_b \to +\infty$. Interestingly, when ϕ_b decreases to negative infinity, the bubble volatility also decreases to zero. However, since in this case the MSV solution is E-unstable, the adaptive learning perspective argues against the relevance of this case: restriction attention to values of ϕ_b for which the solution is learnable, increasing ϕ_b reduces bubble volatility.

The results above show that the volatilities of the old and aggregate bubbles are proportional to the volatility of new bubble innovations, which are the only source of uncertainty. By contrast, for the sunspot solution in Galí (2014) (see (15) here), innovations in old bubbles are another source of uncertainty that can drive the movements of the aggregate bubble independent of fundamentals. This is an appealing feature, though both sources of uncertainty are not observable and hardly testable.

Figure 1 presents the relation between ϕ_b and the volatilities of the old and aggregate bubbles for the MSV solution. We choose the same parameter values as in Galí (2014) by setting $\beta = 1$, $\epsilon = 6$, $U = 0.175$. These values imply $B_s = 0.1$, $B_u = 0.1458$, and $\mathcal{M} = 1.2$. While Galí (2014) studies equilibria around the stable bubbly steady state $B_s = 0.1$, we focus on the solution around the unstable bubbly steady state $B_u = 0.1458$. Galí's result is illustrated in Figure 2 of his paper, which shows that the bubble volatility increases with $\phi_b > 0$.

To understand the intuition behind Figure 1, we consider the economy's responses to an exogenous positive bubble shock to u_t. We first use equations (2), (6), (7), and (8) to derive the log-linearized asset pricing equation

$$q_t = E_t b_{t+1} - r_t, \tag{20}$$

which says that total bubble is equal to the future old bubble discounted by r_t. Using this equation and (19), we see that the old bubble satisfies the asset pricing equation

$$b_t = \frac{1}{R} E_t b_{t+1} - \frac{1}{R} r_t - \frac{(1-R)}{R} u_t. \tag{21}$$

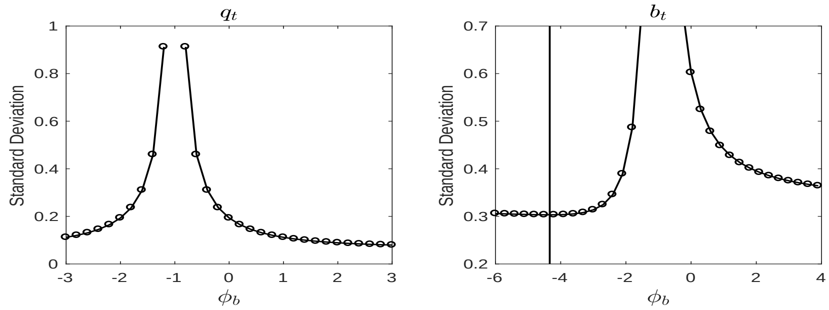

FIGURE 1. MONETARY POLICY AND BUBBLE VOLATILITY. THE VERTICAL LINE ON THE RIGHT PANEL INDICATES THE VALUE OF ϕ_b THAT MINIMIZES THE STANDARD DEVIATION OF THE OLD BUBBLE. THE PARAMETER VALUES ARE $\beta = 1$, $\epsilon = 6$, $U = 0.175$, $\phi_\pi = 2$, $\rho = 0.8$, AND $\sigma_e^2 = 0.01$. WE FOCUS ON THE UNSTABLE BUBBLY STEADY STATE WITH $B = 0.1458$.

Solving forward shows that the old bubble is equal to the (negative) discounted value of future real interest rates and new bubbles. Since $0 < R < 1$, new bubbles $\{u_t\}$ act as negative dividends. An increase in u_t has a direct effect of lowering b_t and an indirect effect through the change in the interest rate r_t. Due to the endogenous response of r_t, a unique forward-looking solution for b_t exists as shown in Proposition 2, even when $0 < R < 1$. In contrast to Galí (2014), b_t is a jump variable and responds to shocks on impact like any asset prices.

The impact of monetary policy on asset bubbles q_t and b_t is transmitted through the real interest rate r_t, which in turn depends on the size of bubbles b_t. Thus we need to understand the dynamic responses of r_t for different values of ϕ_b. In Appendix C we show that

$$r_t = (R-1)\left[\frac{(\epsilon B+1)(\rho - R)}{\chi - \rho} + 1\right]\rho u_{t-1} + \frac{\phi_b \rho (R-1)(\epsilon B + 1)}{(\phi_b + 1)(\chi - \rho)} e_t. \tag{22}$$

When $\rho = 0$, $r_t = 0$ and $b_t = e_t\,(R-1)\,/R$ by Proposition 2. It follows from (20) that $q_t = 0$. The intuition is that the impact of a positive new bubble shock on the aggregate bubble is exactly offset by a negative response of the old bubble so that the size of the aggregate bubble does not change. Thus the value of ϕ_b does not affect the real interest rate by the monetary policy rule in (10) and hence it does not affect bubble dynamics.

Figure 2 presents the impulse response functions for $\{b_t\}, \{q_t\}, \{r_t\}$, and $\{\pi_t\}$ given a 1% shock to e_0 in period 0 for $\rho = 0.8$. When monetary policy does not respond to bubbles ($\phi_b = 0$), a positive shock to expand the new bubble u_0 at date 0 crowds out the size of old bubbles b_0 and dampens the aggregate bubble q_0, but r_0 does not change, as shown in equations (16), (19), and (22).

When $\phi_b > 0$, the central bank will cut the interest rate according to the interest rate rule as q_0 and b_0 decline and hence the fall of the old and aggregate bubbles is mitigated by (21). For this channel to work we need $\rho \in (0,1)$ so that the term related to e_t in (22) is negative as $R \in (0,1)$ and $\chi > 1$. Both $\rho > 0$ and the forward-looking solution (initial changes of aggregate bubbles) are important for a leaning-against-the-wind policy to lower bubble volatility in response to a bubble shock e_0. For a larger $\phi_b > 0$, the mitigation effect is stronger so that aggregate and old bubbles respond less to the bubble shock. This explains why the volatilities of q_t and b_t decrease with $\phi_b > 0$ as illustrated in Figure 1.

By contrast, for Galí's backward-looking solution, we rewrite (21) as

$$b_{t+1} = Rb_t + r_t + (1-R)\,u_t + \xi_{t+1},$$

where $\xi_t = b_t - E_{t-1}b_t$ is a sunspot shock and b_0 is predetermined. Either a positive bubble shock or a positive sunspot shock raises the size of future bubbles without changing the initial size b_0. A leaning-against-the-wind policy with $\phi_b > 0$ will raise the interest rate r_t so that future bubbles will grow even faster. This explains why such a policy will raise bubble volatility in Galí (2014).

For our forward-looking solution, Figure 2 shows that q_t and b_t fall on impact and then gradually rise to their steady state values. Their dynamics for different values of ϕ_b differ only in the initial period. This can be seen from equations (16), (19), and (22) because $R \in (0,1)$, $e_t = 0$ for $t > 0$, and u_t is an AR(1) process with persistence $\rho > 0$. The effect of ϕ_b is only on the terms related to the temporary shock e_t.

When $\phi_b < 0$, the old and aggregate bubbles may rise on impact in response to a positive bubble shock. When the central bank cuts the interest rate to encourage bubbles, this effect may dominate the direct negative effect of the rise in the new bubble on the old bubble as shown in equation (21). As shown in Figure 2, when ϕ_b decreases from -2 to -5, the old and aggregate bubbles are dampened and the fall of interest rate is also mitigated. If bubbles expanded, the central bank would cut the interest rate more, which in turn would encourage bubbles further. This positive feedback effect might make the bubble explode.

Since firms adjust prices one period in advance before shocks are realized, the inflation rate π_t is predetermined. Thus it does not respond to the bubble shock on impact. As shown in Figure 2, it may rise or fall in the second period depending on the value of ϕ_b. In Appendix C we show that the inflation rate around the unstable bubbly steady state is given by

$$\pi_t = \frac{\rho(R-1)[\rho(\epsilon B+1)+(1+\phi_b)(\beta\epsilon BR-\rho)]}{\phi_\pi(\chi-\rho)}u_{t-1}.$$

If $\phi_b = 0$, the inflation rate falls in the second period because $R \in (0,1)$ and $\chi > 1$. The central bank can stabilize inflation by two strategies: First, it can set ϕ_π at an arbitrary large value and set ϕ_b at a finite value. Second, it can set ϕ_π at a finite value and set $\phi_b = \rho(\epsilon B+1)/(\rho - \beta\epsilon BR) - 1$.

In Galí's (2014) model inflation is not a source of welfare losses given synchro-

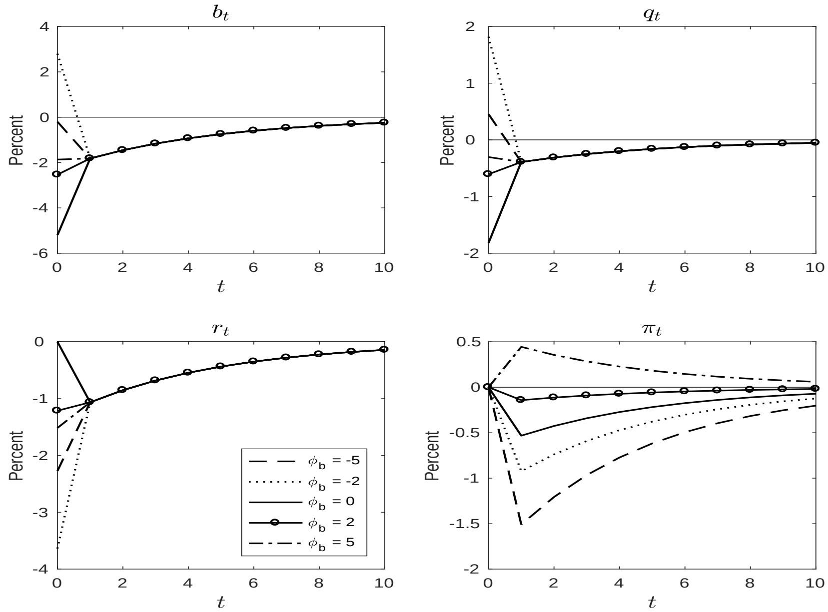

FIGURE 2. IMPULSE RESPONSES TO A NEW BUBBLE SHOCK. THIS FIGURE PLOTS THE IMPULSE RESPONSE FUNCTIONS FOR A ONE PERCENT POSITIVE NEW BUBBLE SHOCK, IN PERCENTAGE DEVIATION FROM THE STEADY STATE. THE PARAMETER VALUES ARE $\beta = 1$, $\epsilon = 6$, $U = 0.175$, $\phi_\pi = 2$, $\rho = 0.8$, AND $\sigma_e^2 = 0.01$. WE FOCUS ON THE UNSTABLE BUBBLY STEADY STATE WITH $B = 0.1458$.

nized price-setting and an inelastic labor supply. Thus it is not optimal for the central bank to stabilize inflation. To study optimal monetary policy, we follow Galí (2014) to take the unconditional mean of an agent's lifetime utility as a welfare criterion. In a neighborhood of a steady state, we can derive the second-order approximation to the mean:

$$E\left[\ln C_{1,t} + \beta \ln C_{2,t+1}\right] \simeq \ln C_1 + \beta \ln C_2 - \frac{1}{2}\left(Var\left(c_{1,t}\right) + Var\left(c_{2,t}\right)\right).$$

By the resource constraint $C_{1,t} + C_{2,t} = 1$, $Var\left(c_{1,t}\right)$ is proportional to $Var\left(c_{2,t}\right)$. Thus the optimal monetary policy that maximizes welfare will minimize the variance of

$$c_{2,t} = (1 - \Gamma)\, d_t + \Gamma b_t,$$

where $\Gamma = \epsilon B/(\epsilon B + 1)$.

In Appendix C we show that

$$d_t = \frac{\chi(R-1)[\phi_b\left(\rho - \epsilon B \beta R\right) - \epsilon B(\beta R + \rho)]}{\beta R^2(1+\phi_b)(\chi - \rho)} e_t.$$

Thus minimizing the volatility of dividends calls for setting

$$\phi_b = \frac{\epsilon B\left(\beta R+\rho\right)}{\epsilon B\beta R-\rho}.$$

However this policy would raise the volatility of the old bubble because it is minimized at a different value given in (18). Thus optimal monetary policy trades off between the volatility of dividends and the volatility of the old bubble.

Note that b_t and d_t are also correlated. In Appendix C we derive that

$$Var(c_{2,t}) = \left(\frac{\epsilon B\rho(R-1)}{\chi-\rho}\right)^2 (1-\rho^2)^{-1}\sigma_e^2 + \left[\frac{(R-1)\rho(\phi_b-\epsilon B)}{\beta R(1+\phi_b)(\chi-\rho)}\right]^2 \sigma_e^2.$$

From this equation we can show that the optimal coefficient is given by $\phi_b = \epsilon B > 0$ for $\rho \neq 0$. Thus the leaning-against-the-wind policy is optimal. Moreover the optimal coefficient increases with the size of the bubble. This property is in contrast with Figure 4 of Galí (2014), which shows that the optimal coefficient ϕ_b is positive for a small size of bubbles and becomes negative for a sufficiently large size of bubbles.

Figure 3 illustrates the relation between ϕ_b and $Var\left(c_{2,t}\right)$ and the relation between ϕ_b and $Var\left(d_t\right)$. The welfare loss is minimized at $\phi_b = 0.875$.

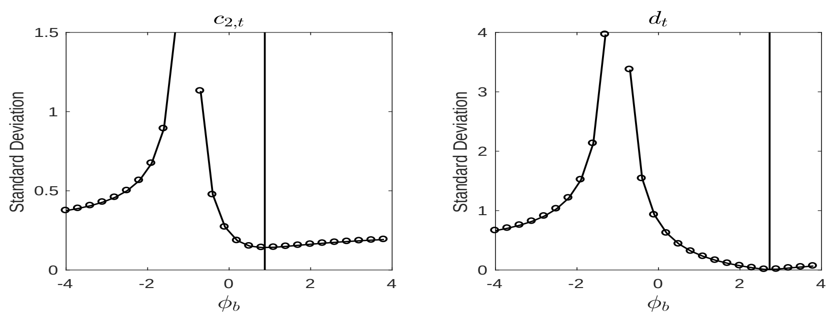

Figure 3. Monetary Policy and Welfare. The vertical lines indicate the values of ϕ_b that minimize the standard deviation of consumption and dividend respectively. The parameter values are $\beta = 1$, $\epsilon = 6$, $U = 0.175$, $\phi_\pi = 2$, $\rho = 0.8$, and $\sigma_e^2 = 0.01$. We focus on the unstable bubbly steady state with $B = 0.1458$.

III. Conclusion

In this paper we have shown that Galí's (2014) counterintuitive results are driven by his choice of a backward-looking sunspot solution around a stable bubbly steady state. His model also features a continuum of unstable bubbly steady states, which Pareto dominate the corresponding stable bubbly steady states.

We extend his model to incorporate persistent bubble shocks. When deriving the unique forward-looking MSV solution around an unstable bubbly steady state, we obtain results that are consistent with the conventional views. We apply learning as a selection device to select steady state and equilibrium. We find that the unstable bubbly steady state and the associated MSV equilibrium are E-stable under optimal monetary policy. But the stable bubbly steady state and the associated sunspot equilibrium are not E-stable under optimal monetary policy.

In an infinite-horizon framework without recurrent creation of new bubbles, Miao and Wang (2018) prove that the economy has two steady states. The local equilibrium around the bubbly steady state is unique and the local equilibrium around the bubbleless steady state is indeterminate of degree one. We conjecture that learning will select the bubbly steady state and the associated forward-looking solution as in this paper. Miao, Wang, and Xu (2015) and Dong, Miao, and Wang (2017) incorporate recurrent bubbles and show that the economy has a continuum of bubbly steady states as in Galí (2014). However, they are unable to prove the stability of these steady states analytically due to the complexity of their multi-dimensional equilibrium systems. In contrast to Galí (2014), their numerical results indicate that each bubbly steady state is a saddle point and the local equilibrium around each bubbly steady state is unique. We suspect that the difference in results may be due to the difference in the infinite-horizon and overlapping-generations frameworks. Further theoretical research is needed to understand this issue.

Appendix

A. Deriving Equilibrium Bubble Dynamics

As in Galí (2014), the log-linearize equilibrium system consists equations (19), (20), and

$$0 = c_{1,t} + \beta R c_{2,t}, \tag{A1}$$

$$c_{1,t} = E_t c_{2,t+1} - r_t, \tag{A2}$$

$$c_{2,t} = (1-\Gamma) d_t + \Gamma b_t, \tag{A3}$$

$$E_{t-1} w_t = E_{t-1} d_t = 0, \tag{A4}$$

$$r_t = \phi_\pi \pi_t + \phi_b q_t. \tag{A5}$$

Combining (A1), (A2), and (A3) we derive

$$\begin{aligned} r_t &= (1-\Gamma) E_t d_{t+1} + \Gamma E_t b_{t+1} + \beta R((1-\Gamma) d_t + \Gamma b_t) \\ &= \Gamma E_t b_{t+1} + \beta R((1-\Gamma) d_t + \Gamma b_t), \end{aligned}$$

where we have used $E_t d_{t+1} = 0$ by (A4) in the second equality.

Combining the equation above with (19) and (20) yields

$$r_t = \Gamma(r_t + Rb_t + (1-R)u_t) + \beta R((1-\Gamma)d_t + \Gamma b_t).$$

We substitute $\Gamma = \epsilon B/(\epsilon B + 1)$ into the equation above to obtain

$$r_t = \epsilon BR(1+\beta)b_t + \epsilon B(1-R)u_t + \beta R d_t. \tag{A6}$$

Taking expectations conditional on information at time $t-1$ yields

$$E_{t-1}r_t = \epsilon BR(1+\beta)E_{t-1}b_t + \epsilon B(1-R)E_{t-1}u_t, \tag{A7}$$

where we have used $E_{t-1}d_t = 0$. We use equation (A7) and interest rate rule (A5) to derive

$$\begin{aligned} r_t - E_{t-1}r_t &= \phi_\pi(\pi_t - E_{t-1}\pi_t) + \phi_b(q_t - E_{t-1}q_t) \\ &= \phi_b(q_t - E_{t-1}q_t) \\ &= \phi_b R(b_t - E_{t-1}b_t) + \phi_b(1-R)(u_t - E_{t-1}u_t), \end{aligned} \tag{A8}$$

where the second equality follows from $\pi_t = E_{t-1}\pi_t$ due to price stickiness and we use (19) to substitute for q_t to derive the third equality.

Using (A7) and (A8) we derive

$$\begin{aligned} r_t &= r_t - E_{t-1}r_t + E_{t-1}r_t \\ &= \phi_b R b_t + (\epsilon B(1+\beta) - \phi_b)RE_{t-1}b_t + \phi_b(1-R)u_t + (\epsilon B - \phi_b)(1-R)E_{t-1}u_t. \end{aligned}$$

Now we substitute the equation above into (20) and use (19) to derive

$$\begin{aligned} E_t b_{t+1} &= Rb_t + (1-R)u_t \\ &\quad + \phi_b R b_t - (\phi_b - \epsilon B(1+\beta))RE_{t-1}b_t + \phi_b(1-R)u_t - (\phi_b - \epsilon B)(1-R)E_{t-1}u_t \\ &= (\phi_b + 1)Rb_t - (\phi_b - \epsilon B(1+\beta))RE_{t-1}b_t + (\phi_b+1)(1-R)u_t - (\phi_b - \epsilon B)(1-R)E_{t-1}u_t. \end{aligned}$$

We then obtain (13). Q.E.D.

B. Proofs

Proof of Proposition 1

Conjecture that the solution takes the form

$$b_t = \varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_2 e_t + \varphi_3 e_{t-1} + \varphi_4 \xi_t + \varphi_5 \xi_{t-1},$$

where $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$, and φ_5 are coefficients to be determined. Substituting this solution into (13) yields

$$\begin{aligned}
b_t \quad = \quad & \frac{1}{R(\phi_b+1)}\left[\varphi_0 b_t + \varphi_1 u_{t-1} + \varphi_3 e_t + \varphi_5 \xi_t\right] \\
& + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\left(\varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_3 e_{t-1} + \varphi_5 \xi_{t-1}\right) \\
& + \frac{R-1}{R}\left(\rho u_{t-1} + e_t\right) + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\rho u_{t-1}.
\end{aligned}$$

That is,

$$\begin{aligned}
b_t \quad = \quad & \frac{1}{R(\phi_b+1)}[\varphi_0\left(\varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_2 e_t + \varphi_3 e_{t-1} + \varphi_4 \xi_t + \varphi_5 \xi_{t-1}\right) \\
& + \varphi_1\left(\rho u_{t-2} + e_{t-1}\right) + \varphi_3 e_t + \varphi_5 \xi_t] \\
& + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\left(\varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_3 e_{t-1} + \varphi_5 \xi_{t-1}\right) \\
& + \frac{R-1}{R}\left(\rho^2 u_{t-2} + \rho e_{t-1} + e_t\right) + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\left(\rho^2 u_{t-2} + \rho e_{t-1}\right).
\end{aligned}$$

Using the conjectured form for b_t again and matching coefficients, we obtain

(B1)
$$\varphi_0 = \frac{1}{R(\phi_b+1)}\varphi_0^2 + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\varphi_0,$$
(B2)
$$\varphi_1 = \frac{1}{R(\phi_b+1)}\left(\varphi_0\varphi_1 + \rho\varphi_1\right) + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\varphi_1 + \frac{R-1}{R}\rho^2 + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\rho^2,$$
(B3)
$$\varphi_2 = \frac{1}{R(\phi_b+1)}\left(\varphi_0\varphi_2 + \varphi_3\right) + \frac{R-1}{R},$$
(B4)
$$\varphi_3 = \frac{1}{R(\phi_b+1)}\left(\varphi_0\varphi_3 + \varphi_1\right) + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\varphi_3 + \frac{R-1}{R}\rho + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\rho,$$
(B5)
$$\varphi_4 = \frac{1}{R(\phi_b+1)}\left(\varphi_0\varphi_4 + \varphi_5\right),$$
(B6)
$$\varphi_5 = \frac{1}{R(\phi_b+1)}\varphi_0\varphi_5 + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}.$$

There are two solutions for $\varphi_0 : \varphi_0 = 0$ and

$$\varphi_0 = \chi = R\left(1 + \epsilon B\left(1 + \beta\right)\right).$$

In a neighborhood of the stable bubbly steady state, we have $\chi \in (0,1)$. The only stationary solution must corresponds to $\varphi_0 = \chi$ as Galí (2014) points out. We can then solve for the other coefficients:

$$\varphi_1 = (1 - R)(1 + \epsilon B)\rho,\ \varphi_2 = \frac{\varphi_3 + (R-1)(1+\phi_b)}{R(\phi_b + 1) - \chi},\ \varphi_4 = \frac{\varphi_5}{R(\phi_b + 1) - \chi},$$

and φ_3 and φ_5 are arbitrary numbers. Q.E.D.

Proof of Proposition 2

We take expectations conditional on information at time $t-1$ on both sides of (13) to obtain

$$\begin{aligned} E_{t-1}b_t\left[1 - \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b + 1}\right] &= \frac{1}{R(\phi_b+1)}E_{t-1}b_{t+1} \\ &+ \left[\frac{R-1}{R} + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\right]\rho u_{t-1}. \end{aligned}$$

This implies that

$$E_{t-1}b_t = \frac{1}{R[1+\epsilon B(1+\beta)]}E_{t-1}b_{t+1} - \frac{(1-R)(\epsilon B + 1)}{R(1+\epsilon B(1+\beta))}\rho u_{t-1}.$$

By iterating the equation above forward we can derive

$$\begin{aligned} E_{t-1}b_t &= -\frac{(1-R)(\epsilon B+1)}{R(1+\epsilon B(1+\beta))}\left(\frac{1}{1-\rho/R[1+\epsilon B(1+\beta)]}\right)\rho u_{t-1} \\ &= -\frac{(1-R)(\epsilon B+1)}{\chi - \rho}\rho u_{t-1}, \end{aligned}$$

under the condition $\chi \equiv R[1+\epsilon B(1+\beta)] > 1$. Therefore we also have

$$E_t b_{t+1} = -\frac{(1-R)(\epsilon B+1)}{\chi-\rho}\rho u_t = -\frac{(1-R)(\epsilon B+1)}{\chi-\rho}(\rho^2 u_{t-1} + \rho e_t).$$

Substituting the preceding expressions for $E_t b_{t+1}$ and $E_{t-1}b_t$ into (13), we obtain the rational expectations solution in (16). Q.E.D.

Proof of Proposition 3

We use lifetime utility as the welfare criterion. Define the steady state welfare as

$$W_f \equiv \ln(C_1) + \beta \ln(C_2),$$

where C_1 and C_2 denote the steady-state consumption of a consumer in his young and old. In a steady state we have $C_1 = 1/\mathcal{M} - B$ and $C_2 = 1 - 1/\mathcal{M} + B$. Therefore

$$W_f = \ln(\frac{1}{\mathcal{M}} - B) + \beta \ln(1 - \frac{1}{\mathcal{M}} + B).$$

We can compute

$$\frac{\partial W_f}{\partial B} = \frac{(\frac{1}{\mathcal{M}} - \frac{1}{1+\beta}) - B}{C_1 C_2 (1+\beta)}.$$

Denote $B^* \equiv 1/\mathcal{M} - 1/(1+\beta)$. Note that $B^* > 0$ under the condition $\mathcal{M} < 1+\beta$. This implies that welfare is increasing with B when $B < B^*$. As shown in Galí (2014) Lemma 1, for any $U \in (0, \bar{U})$ the model has two bubbly steady states. Moreover the stable one B_s is always less than the unstable one B_u. Thus to show the welfare is greater at B_u than at B_s, it suffices to show that $B_u < B^*$.

Since B_u is the larger root of equation $H(B,U) = B$, we have

$$B_u = \frac{-(1 + U - \frac{1+\beta}{\mathcal{M}}) + \sqrt{(1 + U - \frac{1+\beta}{\mathcal{M}})^2 - 4(1+\beta)(1 - \frac{1}{\mathcal{M}})U}}{2(1+\beta)}.$$

Therefore

$$B_u - B^* = \frac{(1 - U - \frac{1+\beta}{\mathcal{M}}) + \sqrt{(1 + U - \frac{1+\beta}{\mathcal{M}})^2 - 4(1+\beta)(1 - \frac{1}{\mathcal{M}})U}}{2(1+\beta)}.$$

Note that $1 - U - \frac{1+\beta}{\mathcal{M}} < 0$ by (12). To show $B_u < B^*$, it suffices to show that

$$(1 - U - \frac{1+\beta}{\mathcal{M}})^2 > (1 + U - \frac{1+\beta}{\mathcal{M}})^2 - 4(1+\beta)(1 - \frac{1}{\mathcal{M}})U.$$

This inequality is equivalent to $4(1+\beta)U > 4U$, which holds true since $U, \beta > 0$.

To study E-stability, we rewrite (13) in a general form

$$b_t = \beta_0 E_{t-1} b_t + \beta_1 E_t b_{t+1} + \gamma_0 u_t + \gamma_1 u_{t-1}, \tag{B7}$$

where

$$\beta_0 = \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b + 1}, \ \beta_1 = \frac{1}{R(\phi_b+1)},$$
$$\gamma_0 = \frac{R-1}{R}, \ \gamma_1 = \frac{\rho(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}.$$

We can check that $\chi \equiv R(1+\epsilon B(1+\beta)) = \beta_1^{-1}(1-\beta_0)$. Suppose that the PLM is $b_t = a$. Set $E_{t-1}b_t = E_t b_{t+1} = a$ and $u_t = u_{t-1} = 0$. Then the ALM is $b_t = T(a) = (\beta_0 + \beta_1) a$. By Evans and Honkaphja (2001), the E-stability condition for the steady state $a = 0$ given the ODE $\dot{a} = T(a) - a = (\beta_0 + \beta_1) a - a$ is $\beta_0 + \beta_1 < 1$. Since $\chi > 1$ for the unstable bubbly steady state and $\chi \in (0,1)$ for the stable bubbly steady state, we immediately establish the proposition. Q.E.D.

Proof of Proposition 4

We start with the MSV solution. We write the PLM as

$$b_t = \mu + \varphi_1 u_{t-1} + \varphi_2 e_t,$$

where we include a constant term μ. Stability under learning requires μ convergence of μ to zero. Plugging this equation into (B7) we obtain the ALM with the map $T(\mu, \varphi_1, \varphi_2)$. By Evans and Honkapohja (2001), the E-stability condition is

$$\beta_0 + \beta_1 < 1, \ \beta_0 + \beta_1 \rho < 1. \tag{B8}$$

Using the definition in the proof of Proposition 3, we have

$$\beta_0 + \beta_1 = \frac{1 + R\phi_b - R\epsilon B(1+\beta)}{R(\phi_b+1)} = 1 + \frac{1-\chi}{R(\phi_b+1)},$$
$$\beta_0 + \rho\beta_1 = \frac{\rho + R\phi_b - R\epsilon B(1+\beta)}{R(\phi_b+1)} = 1 + \frac{\rho-\chi}{R(\phi_b+1)}.$$

Since $\chi > 1$ at the unstable bubbly steady state, the E-stability condition for the MSV solution is $\phi_b > -1$.

Now we consider the backward sunspot solution. We write PLM as

$$b_t = \mu + \varphi_1 b_{t-1} + \varphi_2 u_{t-1} + \varphi_3 e_t + \varphi_4 e_{t-1} + \varphi_5 \xi_t + \varphi_6 \xi_{t-1}.$$

Plugging this equation into (B7) we obtain the ALM with the $T-$map. By Evans and Honkapohja (2001), the E-stability condition is $\beta_0 > 1$, $\beta_1 < 0$. Also the stationarity of the solution requires $|\beta_1^{-1}(1-\beta_0)| < 1$. In terms of our model parameters, the E-stability condition is $\phi_b < -1$. Q.E.D.

C. Deriving MSV Equilibrium

From Proposition 2 we have the forward-looking MSV solution for the old bubble:

$$b_t = \frac{(R-1)(\epsilon B+1)}{\chi-\rho}\rho u_{t-1} + \frac{R-1}{R}\left[\frac{\rho(\epsilon B+1)}{(\phi_b+1)(\chi-\rho)}+1\right]e_t. \tag{C1}$$

We use this solution to derive solutions for other variables in the model. By (19) we obtain the solution for q_t:

$$\begin{aligned} q_t &= Rb_t + (1-R)u_t \\ &= (1-R)\left[1-\frac{R(\epsilon B+1)}{\chi-\rho}\right]\rho u_{t-1} + (R-1)\left[\frac{\rho(1+\epsilon B)}{(\phi_b+1)(\chi-\rho)}\right]e_t. \end{aligned} \tag{C2}$$

By (20) we obtain the solution for r_t:

$$\begin{aligned} r_t &= E_t b_{t+1} - q_t \\ &= (R-1)\left[\frac{(\epsilon B+1)(\rho-R)}{\chi-\rho}+1\right]\rho u_{t-1} + \frac{\phi_b\rho(R-1)(\epsilon B+1)}{(\phi_b+1)(\chi-\rho)}e_t. \end{aligned} \tag{C3}$$

By (A5) we obtain the solution for π_t:

$$\pi_t = \frac{(R-1)[(\epsilon B+1)\rho + (\phi_b+1)(\epsilon BR\beta-\rho)]}{\phi_\pi(\chi-\rho)}\rho u_{t-1}.$$

Substituting (C3) and (C1) into (A6) we obtain the solution for d_t:

$$d_t = \frac{\chi(R-1)[\phi_b\rho - \epsilon B(\beta R(1+\phi_b)+\rho)]}{\beta R^2(1+\phi_b)(\chi-\rho)}e_t.$$

By (A3) we obtain the solution for $c_{2,t}$:

$$\begin{aligned} c_{2,t} &= (1-\Gamma)d_t + \Gamma b_t \\ &= \frac{\epsilon B\rho(R-1)}{\chi-\rho}u_{t-1} + \frac{\rho(R-1)(\phi_b-\epsilon B)}{\beta R(1+\phi_b)(\chi-\rho)}e_t. \end{aligned} \tag{C4}$$

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