A Q-Theory Model with Lumpy Investment*

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Abstract

We present an analytically tractable dynamic stochastic general equilibrium model that incorporates micro-level fixed and convex adjustment costs. We provide an explicit characterization of equilibrium dynamics by a system of nonlinear stochastic difference equations. We provide general conditions under which our model features investment lumpiness at the microeconomic level, but aggregate dynamics are isomorphic to those in a Q-theory model without fixed costs. This theoretical result is independent of the specification of the fixed cost distribution and also holds true when firms face persistent idiosyncratic productivity shocks.

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1 Introduction

Recent empirical studies have documented that lumpiness of micro-level capital adjustment is a widespread phenomenon (see, e.g., Doms and Dunne (1998), Cooper et al. (1999) and Cooper and Haltiwanger (2006)). In addition, using a structural estimation model, Cooper and Haltiwanger (2006) show that both fixed and convex capital adjustment costs are important to explain plant-level investment dynamics. Despite the importance of these adjustment costs at the microeconomic level, the standard approach to modeling business cycles is typically to ignore fixed costs and micro-level investment lumpiness. For example, a standard representative-agent and representative-firm real-business-cycle (RBC) model without any adjustment costs may explain aggregate data reasonably well (e.g., Prescott (1986)).

In this paper, we provide a theoretical study using a tractable dynamic stochastic general equilibrium model that incorporates micro-level fixed and convex capital adjustment costs.[1] Following Caballero and Engel (1999), we assume that fixed costs are stochastic and drawn independently and identically from a fixed distribution across firms and over time. The presence of fixed costs delivers micro-level investment lumpiness and results in a generalized (S,s) investment rule similar to that in Caballero and Engel (1999). The idiosyncratic nature of the fixed costs generates firm heterogeneity. In the cross section, some firms decide to make investments and some do not depending on their draws of the fixed costs.

The main contribution of our theoretical study is to provide general conditions such that in the aggregate the model with both fixed and convex adjustment costs is isomorphic to a Q-theory model without fixed costs and with convex adjustment costs only. Note that the curvatures of the convex adjustment cost function in the two models are different. In an example, we show that the curvature of the convex adjustment cost function in this isomorphic model is smaller than that in the original model. This curvature in the isomorphic model is smaller when the extensive margin effect of capital adjustment is larger in the original model. In this case, the aggregate dynamics in the model with fixed costs are more similar to that in the standard RBC model without any adjustment costs. This theoretical result is consistent with the numerical findings reported by Thomas (2002) and Khan and Thomas (2003, 2008).

An implication of our theoretical model is that the standard Q theory developed by Tobin (1969), Abel (1979) and Hayashi (1982) fails to explain microeconomic investment dynamics as in the data. However, it applies to the aggregate because Tobin's Q is a sufficient statistic

[1] Abel and Eberly (1994) provide a continuous-time partial equilibrium model of investment that incorporates both fixed and convex adjustment costs.

to explain the aggregate investment rate. Q theory of investment has been widely applied in dynamic stochastic general equilibrium models and play an important role in (1) resolving the equity premium puzzle (Jermann (1998), Boldrin, et al. (2001)), (2) explaining the saving and investment correlations (Baxter and Crucini (1993)), (3) estimating welfare costs of business cycles (Barlevy (2004)), and (4) providing a propagation and transmission mechanism of real and financial shocks when combined with agency costs (Bernanke et al. (1999) and Kiyotaki and Moore (1997)).[2] Our analysis demonstrates that it may be theoretically coherent to apply the Q theory in the aggregate even though there exists micro-level investment lumpiness. It also shows that using aggregate data alone to calibrate or estimate the curvature of the convex adjustment cost function may lead to a biased estimate.

There are three key conditions for our result to hold. First, we assume that firms have the same constant-returns-to-scale production function. Second, the convex adjustment cost function is linearly homogenous in investment and capital. Third, a firm's fixed capital adjustment costs are proportional to its existing capital stock. Other than these conditions, our result does not depend on the specification of the fixed cost distribution or any particular functional forms of the production function and the convex adjustment cost function.

The first two assumptions are standard in the Q theory of investment discussed nicely by Hayashi (1982) and widely adopted in the dynamic stochastic general equilibrium literature. The third assumption is a convenient normalization assumption so that fixed costs do not become negligible when a firm's size becomes sufficiently large. These three assumptions allow us to exploit the homogeneity property of firm value to derive a closed-form solution for the generalized (S,s) investment rule. Specifically, there is an investment trigger and a target investment rate, both of which are functions of marginal Q and hence respond to aggregate shocks. A firm makes investment if and only if its fixed costs are smaller than the investment trigger. These three assumptions also allow us to derive exact aggregation so that we can represent aggregate equilibrium dynamics by a system of nonlinear stochastic difference equations as in the RBC literature. In particular, the distribution of capital matters only to the extent of its mean. We then prove that the competitive equilibrium is constrained efficient in the sense that if a social planner chooses allocation and investment trigger and target taken firm-level convex and nonconvex adjustment costs as given, then the optimal allocation and investment trigger and target are the same as those in a competitive equilibrium. This result also implies that a recursive equilibrium exists and unique, which provides the theoretic foundation for a recursive

[2] Wang and Wen (2012) show that financial frictions in the form of collateral constraints at the firm level can generate convex adjustment costs at the aggregate level and investment lumpiness at the firm level.

method to solve the model numerically.

Our model is tractable and can be extended to incorporate many state variables such as investment-specific shocks and other types of aggregate shocks. It can also capture micro-level occasional large positive and negative investments. The main limitation of our model is that the adjustment hazard is flat in that it does not depend on firm-specific capital. Thus, our model rules out distributional dynamics and cannot address distributional asymmetry and nonlinearity emphasized by Caballero et al. (1995). To deal with distributional dynamics, one needs to relax the constant-returns-to-scale assumption. The cost of this modeling is that one has to use complicated numerical methods to handle the firm distribution over capital, an infinite dimensional state variable. The numerical methods developed by Dotsey, King, and Wolman (1999) and by Krusell and Smith (1998) are often used in the literature. Using these methods, Thomas (2002) and Khan and Thomas (2003, 2008) show numerically that distributional dynamics are still not important in more complicated models. These methods can be time consuming when there are many additional state variables (e.g., many aggregate shocks).

Nevertheless, our simple model is still rich enough for us to analyze macroeconomic business cycles with the essential feature of micro-level lumpiness, and also is tractable enough for us to analyze theoretically the effects of intensive margin, extensive margin, and general equilibrium price movements, which are important elements emphasized in the literature. In addition, our model can be numerically solved by the standard log-linear approximation method or the higher-order perturbation method. These methods can be implemented easily by standard package such as Dynare.

Importantly, to capture micro-level investment spikes, we extend our model to incorporate idiosyncratic productivity shocks. In this case, firms are heterogeneous ex post because they may experience different productivity shocks. We then show that the investment trigger, the investment target, and marginal Q all depend on idiosyncratic productivity shocks and differ across firms, implying substantial heterogeneity in micro-level investment. We also show that the previous isomorphic result extends to this case.

Our paper is related to Caplin and Leahy (2004). They analyze an equilibrium model of durable goods cycles based on Caplin and Leahy (2006) and provide conditions under which a representative agent model, which ignores discrete adjustment at the microeconomic level, is observationally equivalent to a model that incorporates discrete adjustment. They also provide a mapping between the preference and technology parameters of the two models. While their goal and findings are the same as ours, their model and conditions are quite different from ours.

The key condition in the model of Caplin and Leahy (2004, 2006) is that there is sufficient heterogeneity across agents in the time between purchases so that the echoes from previous cycles effectively disappear. Inspired by Caplin and Leahy (2004, 2006), House (2008) provides an equilibrium model of investment and argues that the near infinite elasticity of investment timing for long-lived capital goods causes the irrelevance of distributional dynamics.

Our paper is also related to the (S,s) literature on pricing that tries to develop tractable equilibrium models to simplify distributional dynamics.[3] To capture the lumpiness of price adjustments, the widely adopted Calvo (1983) approach assumes that firms can adjust prices each period with a fixed probability. This approach is applied by Sveen and Weinke (2007) to study lumpy investment. Our model may be viewed as a micro-founded, state-dependent, Calvo-type model. In our model, each period firms can adjust investment with some probability. But this probability (adjustment rate) depends on the aggregate state and varies over time. In addition, in our extended model with idiosyncratic productivity shocks, both the adjustment rate and the investment target depend on these idiosyncratic shocks. This feature is useful to capture substantial heterogeneity in micro-level investment.

Caplin and Spulber (1986) and Caplin and Leahy (1991, 1997) construct equilibrium models of pricing in which the cross-sectional distribution of relative prices is uniform. In these models the adjustment rate and the adjustment target do not respond to aggregate shocks and are constant over time.

The remainder of the paper proceeds as follows. Section 2 presents the model. Section 3 analyzes equilibrium properties. Section 4 presents the isomorphic result. Section 5 extends the model to incorporate persistent idiosyncratic productivity shocks. Section 6 concludes. An appendix contains proofs.

2 The Model

Consider an infinite-horizon economy. Time is discrete and indexed by $t = 0, 1, 2, ...$ There is a continuum of heterogeneous production units, indexed by j and distributed uniformly over $[0, 1]$. We identify a production unit with a firm or a plant. There is a continuum of identical households, who trade all firms' shares. Each firm is subject to aggregate labor-augmenting productivity shocks. In addition, each firm faces idiosyncratic shocks to fixed adjustment costs of investments. To focus on the implications of fixed costs for business cycles, we abstract from

[3] Embedding a partial equilibrium model similar to Abel and Eberly (1998) in a continuous-time general equilibrium framework, Miao (2008) studies the effect of corporate tax policy on long-run equilibrium in the presence of fixed costs and irreversibility.

long-run growth. It is straightforward to incorporate growth because our model assumptions are consistent with balanced growth. It is also straightforward to incorporate other types of aggregate shocks such as investment-specific shocks.

2.1 Firms

All firms have an identical production technology that combines labor and capital to produce output. Specifically, if firm j owns capital K_t^j and hires labor N_t^j, it produces output Y_t^j according to the production function:

$$Y_t^j = F\left(K_t^j, A_t N_t^j\right), \tag{1}$$

where A_t represents aggregate labor-augmenting technology shocks and follows a Markov process. Normalize its deterministic steady state at $A = 1$. Assume that F satisfies $F_1 > 0$, $F_2 > 0$, $F_{11} < 0$, $F_{22} < 0$, $F_{12} > 0$ and the usual Inada conditions. In addition, it has constant returns to scale.

Each firm j may make investment I_t^j to increase its existing capital stock K_t^j. Investment incurs both nonconvex and convex adjustment costs. As in Uzawa (1969) and Hayashi (1982), capital accumulation follows the law of motion:

$$K_{t+1}^j = (1-\delta)K_t^j + g\left(I_t^j / K_t^j\right) K_t^j, \ K_0^j \text{ given}, \tag{2}$$

where $\delta \in (0,1)$ is the depreciation rate and g represents "convex" adjustment costs, satisfying $g(0) = 0$, $g' > 0$, and $g'' < 0$.[4] Nonconvex adjustment costs are fixed costs that must be paid if and only if the firm chooses to invest. As in Cooper and Haltiwanger (2006), we measure these costs as a fraction of the firm's capital stock.[5] That is, if firm j makes new investment, then it pays fixed costs $\xi_t^j K_t^j$, which is independent of the amount of investment. As will be clear later, this modeling of fixed costs is important to ensure that firm value is linearly homogenous. In addition, this modeling ensures that fixed costs are still non-negligible even when firms become sufficiently large. Following Caballero and Engel (1999), we assume that ξ_t^j is identically and independently (IID) drawn from a distribution with a density function $\phi > 0$ over $[0, \xi_{\max}]$

[4] The concavity of g ensures that the firm's problem has an optimal solution for investment. As discussed by Hayashi (1982), an alternative way of introducing adjustment costs is to assume that these costs reduce profits or production directly (e.g. Lucas (1967)). In this case, the adjustment cost function must be convex in investment. This modeling does not change the key insight of our analysis.

[5] There are several ways to model fixed adjustment costs in the literature. Fixed costs may be proportional to the demand shock (Abel and Eberly (1998)), profits (Caballero and Engel (1999) and Cooper and Haltiwanger (2006)), or labor costs (Thomas (2002) and Khan and Thomas (2003, 2008)).

across firms and over time. These idiosyncratic costs cause firm heterogeneity. Assume that idiosyncratic fixed cost shocks and aggregate productivity shocks are independent.

Each firm j pays dividends to households who are shareholders of the firm. Dividends are given by

$$D_t^j = Y_t^j - w_t N_t^j - I_t^j - \xi_t^j K_t^j \mathbf{1}_{I_t^j \neq 0}, \tag{3}$$

where w_t is the wage rate. Here $\mathbf{1}_{I_t^j \neq 0}$ is an indicator function taking value 1 if $I_t^j \neq 0$, and value 0, otherwise.

Firm j's objective is to maximize cum-dividends market value of equity P_t^j :

$$\max \ P_t^j \equiv E_t \left[\sum_{s=0}^{\infty} \beta^s \frac{\Lambda_{t+s}}{\Lambda_t} D_{t+s}^j \right], \tag{4}$$

subject to (2) and (3). Here, $\beta^s \Lambda_{t+s}/\Lambda_t$ is the stochastic discount factor between period t and $t+s$. We will show later that Λ_{t+s} is a household's marginal utility in period $t+s$.

2.2 Households

All households are identical and have the same utility function:

$$E\left[\sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t\right) \right], \tag{5}$$

where $\beta \in (0,1)$ is the discount factor, and U satisfies $U_1 > 0$, $U_{11} < 0$, $U_2 < 0$, $U_{22} < 0$ and the usual Inada conditions. Each household chooses consumption C_t, labor supply N_t, and share holdings α_{t+1}^j to maximize utility (5) subject to the budget constraint:

$$C_t + \int \alpha_{t+1}^j \left(P_t^j - D_t^j \right) dj = \int \alpha_t^j P_t^j dj + w_t N_t. \tag{6}$$

The first-order conditions are given by

$$\Lambda_t \left(P_t^j - D_t^j \right) = E_t \beta \Lambda_{t+1} P_{t+1}^j, \tag{7}$$

$$U_1\left(C_t, N_t\right) = \Lambda_t, \tag{8}$$

$$-U_2\left(C_t, N_t\right) = \Lambda_t w_t. \tag{9}$$

Equations (7)-(8) imply that the stock price P_t^j is given by the discounted present value of dividends as in equation (4). In addition, Λ_t is equal to the marginal utility of consumption.

2.3 Competitive Equilibrium

The stochastic processes of quantities $\{I_t^j, N_t^j, K_t^j\}_{t\geq 0}$, $\{C_t, N_t\}_{t\geq 0}$, and prices $\{w_t, P_t^j\}_{t\geq 0}$ for $j \in [0,1]$ constitute a sequential competitive equilibrium if the following conditions are satisfied:

(i) Given prices $\{w_t\}_{t\geq 0}$, $\{I_t^j, N_t^j\}_{t\geq 0}$ solves firm j's problem (4) subject to the law of motion (2) with the initial distribution of firm-level capital stocks taken as given.

(ii) Given prices $\left\{w_t, P_t^j\right\}_{t\geq 0}$, $\{C_t, N_t, \alpha_{t+1}^j\}_{t\geq 0}$ maximizes utility in (5) subject to the budget constraint (6).

(iii) Markets clear in that:

$$\alpha_t^j = 1,$$

$$N_t = \int N_t^j dj,$$

$$C_t + \int I_t^j dj + \int \xi_t^j K_t^j \mathbf{1}_{I_t^j \neq 0} dj = \int F\left(K_t^j, A_t N_t^j\right) dj. \tag{10}$$

3 Equilibrium Properties

We start by analyzing a single firm's optimal investment policy, holding prices fixed. We then conduct aggregation and characterize equilibrium aggregate dynamics by a system of nonlinear difference equations. We show that the equilibrium is constrained efficient.

3.1 Optimal Investment Policy

To simplify problem (4), we first solve a firm's static labor choice decision. Let $n_t^j = N_t^j / K_t^j$. The first-order condition with respect to labor yields:

$$f'\left(A_t n_t^j\right) A_t = w_t, \tag{11}$$

where we define $f(\cdot) = F(1, \cdot)$. This equation reveals that all firms choose the same labor-capital ratio in that $n_t^j = n_t = n(w_t, A_t)$ for all j. We can then derive firm j's operating profits:

$$\max_{N_t^j} F\left(K_t^j, A_t N_t^j\right) - w_t N_t^j = R_t K_t^j,$$

where $R_t = f(A_t n_t) - w_t n_t$ is independent of j. Note that R_t also represents the marginal product of capital because F has constant returns to scale.

Let $i_t^j = I_t^j / K_t^j$ denote firm j's investment rate. We can then express dividends in (3) as

$$D_t^j = \left[R_t - i_t^j - \xi_t^j \mathbf{1}_{i_t^j \neq 0}\right] K_t^j,$$

and rewrite (2) as

$$K_{t+1}^j = \left[1 - \delta + g(i_t^j)\right] K_t^j. \tag{12}$$

The above two equations imply that equity value or firm value are linear in capital K_t^j. We thus conjecture that firm value takes the form $V_t^j K_t^j$, where V_t^j depends on the aggregate and idiosyncratic states (K_t, A_t, ξ_t^j) in that

$$V_t^j = V\left(K_t, A_t, \xi_t^j\right),$$

for some function V to be determined. Using this conjecture, we rewrite problem (4) by dynamic programming:

$$V_t^j K_t^j = \max_{i_t^j} \left[R_t - i_t^j - \xi_t^j \mathbf{1}_{i_t^j \neq 0}\right] K_t^j + E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} V_{t+1}^j K_{t+1}^j\right], \tag{13}$$

subject to (12). Substituting equation (12) into equation (13) yields:

$$V_t^j = \max_{i_t^j} \; R_t - i_t^j - \xi_t^j \mathbf{1}_{i_t^j \neq 0} + \left[1 - \delta + g\left(i_t^j\right)\right] E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} V_{t+1}^j\right], \tag{14}$$

where R_t and Λ_t depend on the aggregate state (K_t, A_t) only.

We aggregate each firm's price of capital V_t^j and define the aggregate value of the firm per unit of capital conditional on the aggregate state (K_t, A_t) as

$$\bar{V}_t \equiv \bar{V}\left(K_t, A_t\right) \equiv \int_0^{\xi_{\max}} V\left(K_t, A_t, \xi\right) \phi(\xi) d\xi, \tag{15}$$

for some function $\bar{V}$. Because ξ_t^j is IID across both time and firms and is independent of aggregate shocks, we obtain:

$$E_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}^j\right] = E_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \int_0^{\xi_{\max}} V\left(K_{t+1}, A_{t+1}, \xi\right) \phi(\xi) d\xi\right] = E_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1}\right], \tag{16}$$

Now, we rewrite problem (14) as

$$V\left(K_t, A_t, \xi_t^j\right) = \max_{i_t^j} \; R_t - i_t^j - \xi_t^j \mathbf{1}_{i_t^j \neq 0} + \left[1 - \delta + g\left(i_t^j\right)\right] E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1}\right]. \tag{17}$$

Using this equation, we can characterize a firm's optimal investment policy by a generalized (S,s) rule. In so doing, we first define marginal Q as the (risk-adjusted) present value of a marginal unit of investment:

$$Q_t = E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1}\right]. \tag{18}$$

Since investment becomes productive with a one period delay, marginal Q is equal to the discounted expected value of the firm of an additional unit of capital in the next period. Define average Q as the ratio of ex-dividend firm value and the replacement cost of capital:

$$\frac{1}{K_{t+1}^j} E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} V_{t+1}^j K_{t+1}^j \right] = E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1} \right] = Q_t.$$

Because firm value is linearly homogeneous in capital, Tobin's average Q is equal to the marginal Q (Hayashi (1982)).

Equation (17) reveals that g must be concave to determine optimal i_t^j. In addition, the optimal investment target i_t^j is related to marginal Q (Abel and Eberly (1994)). Without convex adjustment costs, $g(x) = x$ and i_t^j is indeterminate unless one assumes a decreasing-returns-to-scale technology.

Proposition 1 *Firm j's optimal investment policy is characterized by the (S, s) policy in that there is a unique trigger value $\xi_t^* > 0$ such that the firm invests if and only if $\xi_t^j \leq \min\{\xi_t^*, \xi_{\max}\}$. The trigger value ξ_t^* satisfies the equation:*

$$\xi_t^* = g(i_t^j) Q_t - i_t^j, \tag{19}$$

where i_t^j is the optimal target investment rate satisfying:

$$1 = Q_t g'(i_t^j). \tag{20}$$

When $\xi_{t+1}^* \leq \xi_{\max}$, marginal Q satisfies:

$$Q_t = E_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1 - \delta) Q_{t+1} + \int_0^{\xi_{t+1}^*} \left(\xi_{t+1}^* - \xi \right) \phi(\xi) d\xi \right\}. \tag{21}$$

Equation (19) says that, at the value ξ_t^*, the benefit from investment is equal to the fixed cost of investment. The benefit from investment increases with Q_t. Thus, the investment trigger ξ_t^* also increases with Q_t. If $\xi_t^* \geq \xi_{\max}$, then the firm always invests for any level of fixed costs. In the aggregate with a cross section of firms, this means that all firms decide to invest. In the analysis below, we will focus on an interior solution for which $\xi_t^* < \xi_{\max}$ for all t.

Note that the investment trigger ξ_t^* depends on the aggregate state (K_t, A_t) only. It does not depend on the firm-specific state (K_t^j, ξ_t^j). This observation implies that conditional on the aggregate state, the adjustment hazard, $\int_0^{\xi_t^*} \phi(\xi) d\xi$, is a constant. This result is due to our assumptions of competitive markets, constant-returns-to-scale technology, and the IID distribution of ξ_t^j. When the production function has decreasing returns to scale or there is monopoly power, the investment trigger ξ_t^* and the adjustment hazard will depend on the firm-specific capital stock, as discussed in Caballero et al. (1995), Caballero and Engel (1999), and Khan and Thomas (2003, 2008).

Equation (20) implies that all firms choose an identical target investment rate. One way to make investment targets depend on firm-specific characteristics is to introduce a persistent idiosyncratic productivity shock (Khan and Thomas (2008)). We will study this extension in Section 5.[6]

Equation (20) shows that the optimal investment level is positively related to marginal Q if and only if the firm's idiosyncratic shock ξ_t^j is lower than the trigger value ξ_t^*, conditional on the aggregate state (K_t, A_t). When $\xi_t^j > \xi_t^*$, firm j chooses not to invest. This zero investment is unrelated to marginal Q. As a result, investment may not be related to marginal Q in the presence of fixed adjustment costs, a point made by Caballero and Leahy (1996). When $\xi_t^j < \xi_t^*$, firm j's investment jumps discretely to the target level, $g'^{-1}(1/Q_t)$, exhibiting an investment spike.

Equation (21) is a type of asset-pricing equation. Ignoring the integration term inside the conditional expectation operator in equation (21), this equation states that the expected price of capital or marginal Q is equal to the risk-adjusted present value of the marginal product of capital. The integration term in (21) reflects the option value of waiting in the presence of fixed adjustment costs. Intuitively, when the shock $\xi_t^j > \xi_t^*$, it is not optimal to pay fixed costs to make investment. Firms will wait to invest until $\xi_t^j \leq \xi_t^*$ and there is an option value of waiting.

3.2 Aggregation and Equilibrium Characterization

Given the linear homogeneity feature of firm value, we can conduct aggregation tractably. We define aggregate capital $K_t = \int K_t^j dj$, aggregate labor demand $N_t = \int N_t^j dj$, aggregate output $Y_t = \int Y_t^j dj$, and aggregate investment expenditure $I_t = \int I_t^j dj$.

Proposition 2 *The aggregate equilibrium processes* $\{Y_t, N_t, C_t, I_t, K_{t+1}, Q_t, \xi_t^*\}_{t\geq 0}$ *and each firm's target investment rate* $\{i_t\}_{t\geq 0}$ *are characterized by the following system of stochastic difference equations when* $\xi_t^* \in (0, \xi_{\max})$ *:*[7]

$$\xi_t^* = g(i_t)Q_t - i_t, \tag{22}$$

[6] An alternative way is to introduce idiosyncratic investment-specific technology shocks as in Wang and Wen (2012). The analysis is available upon request.

[7] We omit the standard transversality conditions here.

$$1 = Q_t g'(i_t), \tag{23}$$

$$I_t = i_t K_t \left[\int_0^{\xi_t^*} \phi(\xi) d\xi \right], \tag{24}$$

$$K_{t+1} = (1-\delta)K_t + g(i_t) \left[\int_0^{\xi_t^*} \phi(\xi) d\xi \right] K_t, \tag{25}$$

$$Y_t = F(K_t, A_t N_t) = I_t + C_t + K_t \int_0^{\xi_t^*} \xi \phi(\xi) d\xi, \tag{26}$$

$$\frac{-U_2(C_t, N_t)}{U_1(C_t, N_t)} = A_t F_2(K_t, A_t N_t), \tag{27}$$

$$\begin{aligned} Q_t \quad = \quad & E_t \left\{ \frac{\beta U_1(C_{t+1}, N_{t+1})}{U_1(C_t, N_t)} \left[F_1(K_{t+1}, A_{t+1} N_{t+1}) + (1-\delta) Q_{t+1} \right. \right. \\ & \left. \left. + \int_0^{\xi_{t+1}^*} \left(\xi_{t+1}^* - \xi \right) \phi(\xi) d\xi \right] \right\}. \end{aligned} \tag{28}$$

Equations (22) and (23) follows from (19) and (20) because all firms choose the same target investment rate. We derive equations (24) and (25) by aggregating equations (2) and (20). Equation (24) shows that aggregate investment rate I_t/K_t is positively related to marginal Q as predicted by the standard Q-theory. Using equations (22) and (23), we deduce that Tobin's Q is a sufficient statistic to predict the aggregate investment rate I_t/K_t.

Equation (24) also shows that the aggregate investment rate I_t/K_t is determined by two effects: The term i_t represents the *intensive margin effect,* which is the target investment rate if a firm decides to invest. The term $\int_0^{\xi_t^*} \phi(\xi) d\xi$ represents the *extensive margin effect,* which gives the adjustment rate (the fraction of adjusting firms) or the adjustment hazard. Gourio and Kashyap (2007) argue that the impact of investment lumpiness is mainly determined by the extensive margin effect. To see this effect more transparently, we derive the log-linearized equation (24) as follows:

$$\hat{I}_t - \hat{K}_t = \underbrace{\hat{i}_t}_{\text{intensive}} + \underbrace{\frac{\xi^* \phi(\xi^*)}{\int_0^{\xi^*} \phi(\xi) d\xi} \hat{\xi}_t^*}_{\text{extensive}}, \tag{29}$$

where ξ^* is the steady-state investment trigger and $\hat{X}_t = \ln X_t - \ln X$ denotes the percentage deviation of a variable X_t from its steady state value X. The above equation shows that the impact of the extensive margin effect is determined by the expression $\xi^* \phi(\xi^*) / \int_0^{\xi^*} \phi(\xi) d\xi$, which is equal to the steady-state elasticity of the adjustment rate with respect to the investment trigger. The magnitude of this elasticity depends on the shape of the fixed cost distribution.

It is equal to η when the fixed cost distribution is given by the power function distribution with density $\phi(\xi) = \frac{\eta \xi^{\eta-1}}{(\xi_{\max})^{\eta}}$, $\eta > 0$. Gourio and Kashyap (2007) argue that one needs the fixed cost distribution to be sufficiently compressed for the extensive margin effect to be large. Our tractable model shows clearly that this feature is not essential.

Equation (26) is the resource constraint. The last term in the equation represents the aggregate fixed adjustment costs. The first equality of equation (26) gives aggregate output using a single firm's production function F. This result is primarily due to the constant returns to scale property of F. The representative household's consumption/leisure choice gives equation (27). Equation (28) is an asset pricing equation for the price of capital Q. It is obtained from equation (21).

We can also use equations (22) and (23) to express ξ_t^* as a function of Q_t only. Substituting this expression into other equations in Proposition 2, we find that aggregate equilibrium dynamics are characterized by a system of stochastic difference equations driven by two endogenous states K_t and Q_t and an exogenous state A_t. In this sense, aggregate dynamics are consistent with the predictions of Q-theory, even though Q-theory does not apply to the firm-level investment in the presence of fixed capital adjustment costs. In Section 4, we will formally show that the economy with fixed costs is isomorphic to another economy without fixed costs and with convex adjustment costs only.

Our equilibrium characterization shows transparently the general equilibrium price feedback effect emphasized by Thomas (2002) and Khan and Thomas (2003, 2008). We derive the log-linearized equations (22) and (28) as follows:

$$\hat{\xi}_t^* = \frac{g}{g - g'i} \hat{Q}_t, \tag{30}$$

$$\hat{Q}_t = \beta E_t \hat{Q}_{t+1} - E_t \hat{r}_{t+1} + \frac{-F_{11} F_2}{F_{21}} \frac{\beta}{Q} E_t \left[\hat{A}_{t+1} - \hat{w}_{t+1} \right], \tag{31}$$

where g, g', F_{11}, F_2, and F_{21} are evaluated at the steady-state values and r_{t+1} is the interest rate satisfying $U_1(C_t, N_t) = E_t[\beta U_1(C_{t+1}, N_{t+1}) r_{t+1}]$. In response to a positive shock to productivity, the marginal product of capital rises and thus capital price Q_t rises. This causes the investment target to rise as revealed by the intensive margin effect in equation (23). In addition, it also induces more firms to invest because the investment trigger rises too as shown in equation (30). This in turn generates an extensive margin effect shown in equation (29). In a general equilibrium model, a positive productivity shock also raises the interest rate and the wage rate, dampening the effect of the rise in the capital price Q_t as shown in equation (31). As a result, both the intensive and the extensive margin effects are weaker in a general equilibrium model than that in a model with fixed prices.

3.3 Constrained Efficiency

Is the competitive equilibrium we studied efficient given that firms face nonconvex adjustment costs? To answer this question, we consider a social planner's problem in which he faces the same investment frictions as individual firms. Suppose the planner selects an investment trigger ξ_t^* and an investment target i_t such that all firms choose the same target i_t when the idiosyncratic fixed adjustment cost shock $\xi_t^j \leq \xi_t^*$. We can then aggregate individual firms' capital and investments to obtain the resource constraint (26) and the capital accumulation equation (25). The social planner's problem is to maximize the representative agent's utility (5) subject to these two constraints:

$$\max_{C_t, N_t, K_{t+1}, i_t, \xi_t^*} \quad E\left[\sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t\right)\right]$$

subject to (25) and

$$C_t + i_t K_t \int_0^{\xi_t^*} \phi(\xi) d\xi + K_t \int_0^{\xi_t^*} \xi \phi(\xi) d\xi = F\left(K_t, A_t N_t\right). \tag{32}$$

Proposition 3 *Suppose that ϕ is differentiable. Given the assumptions on the technology and preferences stated in Section 2, the competitive equilibrium allocation and the investment trigger characterized in Proposition 2 are constrained efficient in the sense that they are identical to those obtained by solving a social planner's problem. In addition, the solution is unique.*

The assumptions on the preferences and technology given before ensure that the social planner's problem is a concave problem and hence it has a unique solution for the aggregate allocation despite the existence of micro-level nonconvexity. Proposition 3 is important because we can use it to establish the existence of a recursive equilibrium for our economy by a standard argument as in Stokey and Lucas (1989). As a result, it provides the theoretic foundation for applying a recursive method to solve our model numerically. Lucas and Prescott (1971) establish a similar result for an industry equilibrium model with convex adjustment costs. To the best of our knowledge, no similar result has been proven in models with micro-level fixed costs, e.g., the models in Khan and Thomas (2003, 2008) or Bachmann et al. (2008). In addition, they have not established whether or not a recursive equilibrium exists for these models.

4 Isomorphism to a Q-Theory Model

As we discussed in Section 3.2, the aggregate equilibrium dynamics can be characterized by three state variables K_t, Q_t and A_t. This suggests a Q-theory model may be able to explain aggregate dynamics. In this section, we formally prove that the economy with both fixed and convex adjustment costs is isomorphic to another economy without fixed costs, but with convex adjustment costs only. Importantly, the convex adjustment cost functions in the two economies are different.

To establish this isomorphism result, we define the total investment expenditure as:

$$X_t = I_t + K_t \int_0^{\xi_t^*} \xi\phi(\xi)d\xi. \tag{33}$$

We shall construct an aggregate adjustment cost function G such that it satisfies:

$$K_{t+1} = (1-\delta)K_t + G(X_t/K_t)K_t, \tag{34}$$

and $G' > 0$ and $G'' < 0$.

To this end, we define

$$x_t \equiv \frac{X_t}{K_t} = i_t \int_0^{\xi_t^*} \xi\phi(\xi)d\xi + \int_0^{\xi_t^*} \xi\phi(\xi)d\xi, \tag{35}$$

where we have used equations (33) and (24). By equations (22) and (23), we deduce that i_t and ξ_t^* are functions of Q_t. Using these two equations and taking differentiation in (35) with respect to Q_t, we obtain:

$$\begin{aligned} \frac{\partial x_t}{\partial Q_t} &= \frac{\partial i_t}{\partial Q_t} \int_0^{\xi_t^*} \phi(\xi)d\xi + \left[\xi_t^*\phi(\xi^*) + i_t\phi(\xi^*)\right] \frac{\partial \xi_t^*}{\partial Q_t} \\ &= -\frac{g'(i_t)}{g''(i_t)} \frac{1}{Q_t} + \left[\xi_t^*\phi(\xi^*) + i_t\phi(\xi^*)\right] \left(\left[g'(i_t)Q_t - 1\right] \frac{\partial i_t}{\partial Q_t} + g(i_t) \right) \\ &= -\frac{g'(i_t)}{g''(i_t)} \frac{1}{Q_t} + \left[g(i_t)\right]^2 Q_t \phi(\xi^*) > 0, \end{aligned} \tag{36}$$

where we have used the fact that $g' > 0$ and $g'' < 0$. Thus, we can write Q_t as a function of x_t. This allows us to express i_t and ξ_t^* as implicit functions of x_t in that $i_t = i(x_t)$ and $\xi_t^* = \xi^*(x_t)$ where $i(\cdot)$ and $\xi^*(\cdot)$ are some functions.

We now define a function G as

$$G(x) = g(i(x)) \int_0^{\xi^*(x)} \phi(\xi)d\xi. \tag{37}$$

We then obtain our isomorphic result:

Proposition 4 *Let* $\{Y_t, N_t, C_t, I_t, K_{t+1}, Q_t, \xi_t^*\}_{t\geq 0}$ *be the equilibrium processes for the economy with lumpy investment characterized in Proposition 2. Define* X_t *by (33) and* G *by (37). Then the processes* $\{Y_t, N_t, C_t, X_t, K_{t+1}, Q_t\}_{t\geq 0}$ *constitute an equilibrium for the economy without fixed costs and with the convex adjustment cost function given by* G*, where* X_t *represents aggregate investment for this economy.*

This proposition demonstrates that the equilibrium dynamics of output Y_t, hours N_t, consumption C_t, capital K_t, and capital price Q_t in the economy with both fixed and convex adjustment costs are identical to those in another economy without fixed costs and with convex adjustment costs. However, aggregate investment in the two economies is different. For the isomorphic result to hold, we include aggregate fixed costs,

$$K_t \int_0^{\xi_t^*} \xi \phi(\xi) d\xi,$$

as part of aggregate investment X_t in (33) for the economy with convex adjustment costs only. Since aggregate fixed costs relative to aggregate capital are small (see Gourio and Kashyap (2007)), the aggregate investment rates in the two economies are similar.

Under the assumptions in Proposition 4, our model cannot generate asymmetric responses of aggregate investment or capital stock to a positive versus a negative technology shock. In addition, our model cannot generate skewness and kurtosis of aggregate investment or of small firm returns as in the data (e.g., Caballero and Engel (1999) and Herranz, Krasa and Villamil (2009, 2013)). Note that an important assumption for this result is that the investment trigger ξ_t^* must be in the interior of the support and the shock must be small. When this assumption is violated, our model can generate asymmetric responses given the nonlinear equilibrium system. For example, Miao and Wang (2013) apply our model to study the impact of corporate tax policy on the macroeconomy. They find asymmetric effects of tax changes. The intuition is as follows. When the steady state investment trigger is close to the upper support, a large corporate tax cut may raise the investment trigger such that it hits the upper support. During the transition phase, the investment trigger can be at the upper support for several periods. But an identical magnitude of tax increase may not cause the investment trigger to hit the lower support.

To further explore the empirical implications of our isomorphic result, we consider two examples of adjustment cost functions widely used in the literature. First, we set

$$g\left(\frac{I}{K}\right) = \frac{\psi}{1-\theta}\left(\frac{I}{K}\right)^{1-\theta}, \tag{38}$$

where $\psi > 0$ and $\theta \in (0,1)$. This specification is used by Barlevy (2004) and Jermann (1998), for example. We can then derive firm j's target investment rate:

$$i_t^j = (\psi Q_t)^{\frac{1}{\theta}}. \tag{39}$$

We also consider the power function distribution for the fixed cost with density $\phi(\xi) = \frac{\eta \xi^{\eta-1}}{(\xi_{\max})^{\eta}}$, $\eta > 0$. We can then use (37) to derive the aggregate convex adjustment cost function:

$$G(x) = \varphi\, x^{1-\frac{\theta}{1+\eta}}, \tag{40}$$

where

$$\varphi = \frac{\psi}{1-\theta}\left(\frac{1}{1+\frac{\eta}{\eta+1}\frac{\theta}{1-\theta}}\right)^{\frac{\eta+1-\theta}{\eta+1}}\left(\frac{\theta}{1-\theta}\frac{1}{\xi_{\max}}\right)^{\frac{\theta\eta}{\eta+1}}. \tag{41}$$

In this case, the shape parameter η of the fixed cost distribution enters the curvature of the aggregate adjustment cost function G. In particular, it makes this curvature smaller. As we show in Section 3.2, η also represents the strength of the extensive margin, i.e., the steady-state elasticity of the adjustment rate with respect to the investment trigger. When η is larger, the extensive margin effect is larger. It also implies that the curvature of the convex adjustment cost function in the isomorphic Q-theory model is smaller. When η is sufficiently large, $G(x)$ approaches a linear function and hence the equilibrium dynamics in the lumpy investment model approaches to those in a RBC model without any adjustment costs.

Our isomorphic result also implies that using aggregate investment data to estimate or calibrate the convex adjustment cost function may be misleading. If we acknowledge the fact that both fixed and convex adjustment costs appear at the plant level, then both parameters η and θ enter the curvature of the aggregate adjustment cost function and hence these two parameters θ and η cannot be identified using aggregate data alone.

One undesirable feature of the specification in (38) is that investment cannot be negative. To allow for negative investment, consider the following specification used by Lucas and Prescott (1971) and Christiano, Eichenbaum and Rebelo (2010):

$$g(x) = x - \frac{\psi}{2}(x-\delta)^2 + \frac{\psi}{2}\delta^2,$$

where $g'(x) > 0$ for $x < 1/\psi + \delta$. In this case, we can derive firm j's target investment rate:

$$i_t^j = \delta + \frac{1}{\psi}\left(1 - \frac{1}{Q_t}\right).$$

If $Q_t < 1/(1+\psi\delta)$, then $i_t^j < 0$. Thus, this specification allows for both negative and positive investment. Given a specification for the fixed cost distribution, we can also derive the aggregate adjustment cost function G.

5 Extension: Firm-Level Idiosyncratic Productivity Shocks

In the previous analysis, the investment target depends on the aggregate state only, but does not depend on firm-specific characteristics. This implies that if firms choose to invest, then they all will invest at the same target level. To be more consistent with the micro-level evidence on investment spikes documented by Cooper and Haltiwanger (2006), we follow Khan and Thomas (2008) and introduce firm-level idiosyncratic productivity shocks into the model presented in Section 2. In this case, the investment target will depend on the idiosyncratic productivity shocks. We will show that our isomorphic result carries over to this extension.

Suppose that firm j's production function is given by

$$Y_t^j = Z_t^j F\left(K_t^j, A_t N_t^j\right), \tag{42}$$

where Z_t^j follows a discrete Markov process with a finite state space $\{z_1, z_2, ...z_M\}$. Suppose that Z_t^j is independent across firms and also independent of aggregate shocks A_t and fixed cost shocks ξ_t^j. Let the transition matrix be given by

$$\Pr(Z_{t+1}^j = z_m | Z_t^j = z_\ell) = \rho_{\ell m}. \tag{43}$$

Assume that the idiosyncratic shocks are drawn from the stationary distribution $(\pi_1, \pi_2, ..., \pi_M)$. Because there is no entry and exit, there are π_ℓ firms with the idiosyncratic productivity level z_ℓ each period by a law of large numbers.

As in Section 3.1, we first solve the static labor choice problem:

$$\max_{N_t^j} \; Z_t^j F\left(K_t^j, A_t N_t^j\right) - w_t N_t^j = R_t(Z_t^j) K_t^j,$$

where $R_t(Z_t^j)$ is the rental rate of capital. Because of the constant-returns-to-scale technology, $R_t(Z_t^j)$ does not depend on K_t^j, but depends on Z_t^j. Similar to the analysis in Section 3.1, we conjecture that the value function takes the form $V_t(\xi_t^j, Z_t^j) K_t^j$, where we suppress the aggregate state as an argument in the function V_t. We can then rewrite firm j's dynamic programming problem as

$$V_t(\xi_t^j, Z_t^j) = \max_{i_t^j} \; R_t(Z_t^j) - i_t^j - \xi_t^j \mathbf{1}_{i_t^j \neq 0} + \left[1 - \delta + g(i_t^j)\right] E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} V_{t+1}(\xi_{t+1}^j, Z_{t+1}^j) | \xi_t^j, Z_t^j\right]. \tag{44}$$

Denote $V_{\ell t}(\xi_t) \equiv V_t(\xi_t, z_\ell)$, $R_{\ell t} \equiv R_t\left(z_\ell\right)$, and $\bar{V}_{\ell t} \equiv \int_0^{\xi_{\max}} V_{\ell t}(\xi, z_\ell) \phi(\xi) d\xi$. Define

$$Q_{\ell t} \equiv E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} V_{t+1}(\xi_{t+1}, Z_{t+1}) | \xi_t, Z_t = z_\ell\right]. \tag{45}$$

Since ξ_t^j is IID, we can show that

$$Q_{\ell t} = E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} V_{t+1}(\xi_{t+1}, Z_{t+1}) | Z_t = z_\ell \right] = E_t \left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} \sum_{m=1}^{M} \rho_{\ell m} \bar{V}_{m,t+1} \right]. \tag{46}$$

It follows that marginal Q depends on idiosyncratic productivity shocks and hence firms with different productivity shocks have different marginal Q.

It follows from (44) that

$$V_{\ell t}(\xi) = \max_{i_t} \ R_{\ell t} - i_t - \xi \mathbf{1}_{i_t^j \neq 0} + [1 - \delta + g(i_t)] \, Q_{\ell t}, \tag{47}$$

for $\ell = 1, 2, ..., M$. The first-order condition with respect to i_t gives the optimal investment target:

$$1 = g'(i_{\ell t}) Q_{\ell t}. \tag{48}$$

This target level also depends on idiosyncratic productivity shocks. Intuitively, a firm hitting by a high idiosyncratic productivity shock invests at a high level. The firm with productivity shock z_ℓ will choose to invest at the target $i_{\ell t}$ if and if

$$R_{\ell t} + (1 - \delta) Q_{\ell t} - i_{\ell t} - \xi + g(i_{\ell t}) Q_{\ell t} \geq R_{\ell t} + (1 - \delta) Q_{\ell t}, \tag{49}$$

or if and only if

$$\xi \leq \xi_{\ell t}^* \equiv g\left(i_{\ell t}\right) Q_{\ell t} - i_{\ell t}. \tag{50}$$

We shall focus on the case where $\xi_{\ell t}^* \leq \xi_{\max}$. Clearly, the trigger level $\xi_{\ell t}^*$ depends on idiosyncratic productivity shocks. We can now rewrite equation (47) as

$$V_{\ell t}(\xi) = \left\{ \begin{array}{cc} R_{\ell t} + (1 - \delta) Q_{\ell t} & \text{if } \xi > \xi_{\ell t}^* \\ R_{\ell t} + (1 - \delta) Q_{\ell t} + \xi_{\ell t}^* - \xi & \text{if } \xi \leq \xi_{\ell t}^* \end{array} \right. . \tag{51}$$

And hence we obtain:

$$\bar{V}_{\ell t} = R_{\ell t} + (1 - \delta) Q_{\ell t} + \int_0^{\xi_{\ell t}^*} \left[\xi_{\ell t}^* - \xi \right] \phi(\xi) d\xi, \ \ell = 1, ..., M. \tag{52}$$

Substituting this equation into (46) yields:

$$Q_{\ell t} = E_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \sum_{m=1}^{M} \rho_{\ell m} \left\{ R_{m,t+1} + (1 - \delta) \, Q_{m,t+1} + \int_0^{\xi_{m,t+1}^*} \left[\xi_{m,t+1}^* - \xi \right] \phi(\xi) d\xi \right\}. \tag{53}$$

Because firms differ in their idiosyncratic productivity levels, for firms with idiosyncratic productivity level z_ℓ, we define their aggregate capital $K_{\ell t} = \int_{Z_t^j = z_\ell} K_t^j dj$, aggregate labor demand $N_{\ell t} = \int_{Z_t^j = z_\ell} N_t^j dj$, aggregate output $Y_{\ell t} = \int_{Z_t^j = z_\ell} Y_t^j dj$, and aggregate investment $I_{\ell t} = \int_{Z_t^j = z_\ell} I_t^j dj$. Aggregating these variables, we can derive the economy-wide aggregates. For example, the economy-wide aggregate output level is given by:

$$Y_t = \sum_\ell Y_{\ell t}.$$

Similar to Proposition 2, we can show the following result. We omit its straightforward proof.

Proposition 5 *The aggregate equilibrium processes* $\{Y_t, N_t, C_t, Y_{\ell t}, N_{\ell t}, I_{\ell t}, K_{\ell t}, Q_{\ell t}, \xi_{\ell t}^*\}_{t \geq 0}$ *are characterized by the following system of difference equations:*[8]

$$1 = Q_{\ell t} g'(i_{\ell t}),$$

$$\xi_{\ell t}^* = g(i_{\ell t}) Q_{\ell t} - i_{\ell t},$$

$$I_{\ell t} = i_{\ell t} \left[\int_0^{\xi_{\ell t}^*} \phi(\xi) d\xi \right] K_{\ell t},$$

$$K_{\ell t+1} = \sum_m \rho_{m\ell} \left\{ (1-\delta) K_{mt} + g(i_{mt}) K_{mt} \int_0^{\xi_{mt}^*} \phi(\xi) d\xi \right\}, \tag{54}$$

$$\begin{aligned} Y_t &= \sum_\ell Y_{\ell t} = \sum_\ell z_\ell F\left(K_{\ell t}, A_t N_{\ell t}\right), \\ &= \sum_\ell I_{\ell t} + C_t + \sum_\ell K_{\ell t} \int_0^{\xi_{\ell t}^*} \xi \phi(\xi) d\xi, \end{aligned}$$

$$N_t = \sum_\ell N_{\ell t},$$

$$\frac{-U_2\left(C_t, N_t\right)}{U_1\left(C_t, N_t\right)} = w_t,$$

$$w_t = z_\ell F_2(K_{\ell t}, A_t N_{\ell t}),$$

where $Q_{\ell t}$ *satisfies (53) and*

$$R_{\ell t} = z_\ell F_1(K_{\ell t}, A_t N_{\ell t}).$$

[8] We omit the standard transversality conditions here.

Except for equation (54), other equations in this proposition are easy to interpret. Equation (54) states that the aggregate capital stock for firms with the productivity level z_ℓ is equal to the sum of the aggregate capital stock for firms with the productivity level z_m, $m = 1, ..., M$, in the period that switch to the productivity level z_ℓ. Since the transition probability is equal to $\rho_{m\ell}$, we obtain the weight $\rho_{m\ell}$ in equation (54) by a law of large numbers.

To establish an isomorphic result, we construct an aggregate convex adjustment cost function G as in Section 4. Consider an otherwise identical economy without fixed costs and with convex adjustment cost function G. Let $\{Y_t, N_t, C_t, Y_{\ell t}, N_{\ell t}, I_{\ell t}, K_{\ell t}, Q_{\ell t}, \xi^*_{\ell t}\}_{t\geq 0}$ be the equilibrium processes for the economy with lumpy investment characterized in Proposition 5. Define

$$X_{\ell t} \equiv I_{\ell t} + K_{\ell t} \int_0^{\xi^*_{\ell t}} \xi \phi(\xi) d\xi. \tag{55}$$

Then we can follow a similar proof for Proposition 4 to show that the processes $\{Y_t, N_t, C_t, Y_{\ell t}, N_{\ell t}, X_{\ell t}, K_{\ell t}, Q_{\ell t}, \xi^*_{\ell t}\}_{t\geq 0}$ constitute an equilibrium for the economy without fixed costs and with the convex adjustment cost function G given by (37).[9]

6 Conclusion

We have presented an analytically tractable dynamic stochastic general equilibrium business cycle model that incorporates micro-level fixed and convex adjustment costs. We provide an explicit characterization of equilibrium dynamics by a system of nonlinear stochastic difference equations, which are convenient for numerical solutions. We prove that our model features investment lumpiness at the microeconomic level, but aggregate dynamics are isomorphic to those in a Q-theory model without fixed costs. This result is also valid when firms face persistent idiosyncratic productivity shocks. We prove that the competitive equilibrium is constrained efficient. The key condition for our results is that the production function features constant returns to scale. Although some researchers find empirical evidence of decreasing-returns-to-scale technology (e.g., Cooper and Haltiwanger (2006)), the assumption of constant-returns-to-scale technology has been widely applied in macroeconomics and is convenient for model analysis. This assumption provides a simple benchmark and helps us see the general equilibrium effects and the intensive and extensive margin effects transparently

Our model serves as a theoretical benchmark for understanding the general equilibrium effect of micro-level investment lumpiness on the aggregate dynamics. Our analysis demonstrates that it may be theoretically coherent to apply the Q theory in the aggregate even though there

[9]The proof is omited here and available upon request.

exists micro-level investment lumpiness. It also shows that using aggregate data alone to calibrate or estimate the curvature of the convex adjustment cost function may lead to a biased estimate. Our tractable model may be applied to address many issues in macroeconomics. For example, Miao and Wang (2013) apply our model to study the impact of corporate tax policy. One may integrate our model in a New Keynesian framework as in Sveen and Weinke (2007). One limitation of our model is that it is not suitable for addressing distributional asymmetry and aggregate nonlinearity. To address this issue, it is necessary to relax the assumption of constant returns to scale. In this case, the distribution of capital is a state variable and we are unable to derive a closed-form solution. One has to use a numerical method to approximate the firm distribution of capital.

Appendix

A Proofs

Proof of Proposition 1: From (17), we can show that the target investment level i_t^j satisfies the first-order condition:

$$1 = g'\left(i_t^j\right) E_t\left[\frac{\beta\Lambda_{t+1}}{\Lambda_t}\bar{V}_{t+1}\right]. \tag{A.1}$$

By equation (18), we can derive equation (20). Using this equation, we define $V^a\left(K_t, A_t, \xi_t^j\right)$ as the price of capital when the firm chooses to invest. It is given by

$$V^a\left(K_t, A_t, \xi_t^j\right) = R_t - i_t^j - \xi_t^j + \left[1 - \delta + g(i_t^j)\right] Q_t \tag{A.2}$$

Define $V^n\left(K_t, A_t\right)$ as the price of capital when the firm chooses not to invest. It satisfies:

$$V^n\left(K_t, A_t\right) = R_t + (1-\delta) Q_t, \tag{A.3}$$

which is independent of ξ_t^j. We can then rewrite problem (17) as

$$V\left(K_t, A_t, \xi_t^j\right) = \max\left\{V^a\left(K_t, A_t, \xi_t^j\right),\ V^n\left(K_t, A_t\right)\right\}. \tag{A.4}$$

Clearly, there is a unique cutoff value ξ_t^* given in (19) satisfying the condition:

$$V^a\left(K_t, A_t, \xi_t^*\right) = V^n\left(K_t, A_t\right), \tag{A.5}$$

$$V^a\left(K_t, A_t, \xi_t^j\right) \geq V^n\left(K_t, A_t\right) \text{ if and only if } \xi_t^j \leq \xi_t^*. \tag{A.6}$$

Because the support of ξ_t^j is $[0, \xi_{\max}]$, the investment trigger is given by $\min\left\{\xi_t^*, \xi_{\max}\right\}$.

When $\xi_t^* \leq \xi_{\max}$, we show that

$$\begin{aligned}
\bar{V}_t &= \int_0^{\xi_{\max}} V\left(K_t, A_t, \xi\right)\phi(\xi)d\xi \\
&= \int_{\xi_t^*}^{\xi_{\max}} V^n\left(K_t, A_t\right)\phi(\xi)d\xi + \int_0^{\xi_t^*} V^a\left(K_t, A_t, \xi\right)\phi(\xi)d\xi \\
&= V^n\left(K_t, A_t\right) + \int_0^{\xi_t^*}\left[V^a\left(K_t, A_t, \xi\right) - V^n\left(K_t, A_t\right)\right]\phi(\xi)d\xi.
\end{aligned}$$

We use equations (A.2), (A.3) and (19) to derive

$$\begin{aligned}
V^a\left(K_t, A_t, \xi\right) - V^n\left(K_t, A_t\right) &= g\left(i_t^j\right) Q_t - i_t^j - \xi \\
&= \xi_t^* - \xi.
\end{aligned} \tag{A.7}$$

Using the above two equations, (A.3), and (18), we obtain (21). Q.E.D.

Proof of Proposition 2: From (11), we deduce that all firms choose the same labor-capital ratio n_t. We thus obtain $N_t = n_t K_t$. We then derive

$$\begin{aligned} Y_t &= \int Y_t^j dj = \int F\left(K_t^j, A_t N_t^j\right) dj = \int F\left(1, A_t n_t^j\right) K_t^j dj \\ &= F\left(1, A_t n_t\right) \int K_t^j dj = F\left(1, A_t n_t\right) K_t = F\left(K_t, A_t N_t\right), \end{aligned}$$

which gives the first equality in equation (26). As a result, we use equation (11) and $n_t^j = n_t$ to show

$$A_t F_2(K_t, A_t N_t) = w_t. \tag{A.8}$$

By the constant return to scale property of F, we also have

$$R_t = F_1(K_t, A_t N_t). \tag{A.9}$$

Equation (22) follows from equation (19) and (8). We next derive aggregate investment:

$$I_t = \int I_t^j dj = \int i_t^j K_t^j dj = K_t \int_0^{\xi_t^*} i_t \phi\left(\xi\right) d\xi,$$

where the second equality uses the definition of i_t^j, the third equality uses a law of large numbers and the optimal investment rule (39). We thus obtain (24).

We turn to the law of motion for capital. By definition,

$$K_{t+1} = \left(1 - \delta\right) K_t + \int g\left(i_t^j\right) K_t^j dj.$$

Substituting optimal investment in equation (39) and using equation (24), we obtain (25).

Equation (28) follows from substitution of equations (8) and (A.9) into equation (21). Equation (27) follows from equations (8), (9) and (A.8). Finally, equation (26) follows from a law of large number, the market clearing condition (10), and Proposition 1. Q.E.D.

Proof of Proposition 3: Let λ_t and $\lambda_t Q_t$ be the Lagrange multipliers associated with (32) and (25), respectively. Write the Lagrangian as

$$\begin{aligned} &\max_{C_t, N_t, K_{t+1}, i_t, \xi_t^*} E \sum_{t=0}^{\infty} \beta^t \left\{ U\left(C_t, N_t\right) - \lambda_t Q_t \left(K_{t+1} - \left(1 - \delta\right) K_t - g\left(i_t\right) K_t \int_0^{\xi_t^*} \phi(\xi) d\xi \right) \right. \\ &\left. -\lambda_t \left(C_t + i_t K_t \int_0^{\xi_t^*} \phi(\xi) d\xi + K_t \int_0^{\xi_t^*} \xi \phi(\xi) d\xi - F\left(K_t, A_t N_t\right) \right) \right\}. \end{aligned}$$

The first-order conditions are given by

$$C_t : U_1\left(C_t, N_t\right) = \lambda_t, \tag{A.10}$$

$$N_t : -U_2\left(C_t, N_t\right) = \lambda_t A_t F_2\left(K_t, A_t N_t\right), \tag{A.11}$$

$$i_t : g'\left(i_t\right) Q_t = 1,$$

$$\begin{aligned} K_{t+1} \quad &: \quad \lambda_t Q_t = E_t \beta \lambda_{t+1} \left[F_1\left(K_{t+1}, A_{t+1} N_{t+1}\right) + (1-\delta) Q_{t+1} - \int_0^{\xi_{t+1}^*} \xi \phi(\xi) d\xi \right] \\ &+ E_t \beta \lambda_{t+1} \left(Q_{t+1} g\left(i_{t+1}\right) - i_{t+1}\right) \int_0^{\xi_{t+1}^*} \phi(\xi) d\xi, \end{aligned} \tag{A.12}$$

$$\xi_t^* : \lambda_t Q_t g\left(i_t\right) \phi\left(\xi_t^*\right) K_t - \lambda_t \left[\xi_t^* \phi\left(\xi_t^*\right) + \phi\left(\xi_t^*\right) i_t\right] K_t = 0. \tag{A.13}$$

Given the assumptions on the preferences and technology, we can check that the second-order conditions are also satisfied. Here we verify the second-order condition for ξ_t^* only:

$$\lambda_t K_t \left(Q_t g\left(i_t\right) \phi'\left(\xi_t^*\right) - \phi\left(\xi_t^*\right) - \xi_t^* \phi'\left(\xi_t^*\right) - \phi'\left(\xi_t^*\right) i_t\right) = -\lambda_t K_t \phi\left(\xi_t^*\right) < 0,$$

where we have used the first-order condition (A.13) to derive the equality and the fact that $\lambda_t > 0$ to derive the inequality. Thus, the above first-order conditions give the solution for the constrained efficient allocation and the investment trigger.

We can easily check that the above first-order conditions give the equations in Proposition 2. Thus, the competitive equilibrium allocation and the investment trigger $\{C_t, N_t, K_{t+1}, i_t, \xi_t^*\}$ satisfy the above first-order conditions and hence are constrained efficient. Q.E.D.

Proof of Proposition 4: By the definition of X_t in (33) and (26), we obtain:

$$C_t + X_t = F\left(K_t, A_t N_t\right). \tag{A.14}$$

By the definition of G in (37) and (25), we obtain equation (34). In addition, we can show that

$$\begin{aligned} G(x_t) Q_t - x_t &= Q_t g(i_t) \int_0^{\xi_t^*} \phi(\xi) d\xi - \frac{I_t}{K_t} - \int_0^{\xi_t^*} \xi \phi(\xi) d\xi \\ &= \left[Q_t g(i_t) - i_t\right] \int_0^{\xi_t^*} \phi(\xi) d\xi - \int_0^{\xi_t^*} \xi \phi(\xi) d\xi \\ &= \int_0^{\xi_t^*} \left[\xi_t^* - \xi\right] \phi(\xi) d\xi, \end{aligned} \tag{A.15}$$

where we have used equation (22) and the definition $x_t = X_t / K_t$. Substituting (A.15) into (28), we obtain:

$$\begin{aligned} Q_t \quad = \quad & E_t \left\{ \frac{\beta U_1\left(C_{t+1}, N_{t+1}\right)}{U_1\left(C_t, N_t\right)} \left[F_1\left(K_{t+1}, A_{t+1} N_{t+1}\right) + (1-\delta) Q_{t+1} \right.\right. \\ & \left.\left. + Q_{t+1} G\left(X_{t+1}/K_{t+1}\right) - X_{t+1}/K_{t+1}\right]\right\}. \end{aligned} \tag{A.16}$$

Differentiating (A.15) with respect to Q_t, we obtain:

$$[G'(x_t)Q_t - 1]\frac{\partial x_t}{\partial Q_t} + G(x_t) = \int_0^{\xi_t^*} \phi(\xi)d\xi \frac{\partial \xi_t^*}{\partial Q_t} \tag{A.17}$$

Notice that

$$\frac{\partial \xi_t^*}{\partial Q_t} = [g'(i_t)Q_t - 1]\frac{\partial i_t}{\partial Q_t} + g(i_t) = g(i_t), \tag{A.18}$$

where we have used equation (23). By definition of G in (37), it follows from (A.17) that

$$G'(x_t)Q_t = 1. \tag{A.19}$$

This also implies that $G' > 0$. Compute

$$G''(x_t) = \frac{-1}{Q_t \frac{\partial x_t}{\partial Q_t}} < 0,$$

where we have used (36).

Now, consider the economy without fixed costs and with convex adjustment costs given by G. The equilibrium allocation $\{C_t, N_t, K_{t+1}, X_t\}$ for this economy is Pareto optimal and solves the social planner's problem:

$$\max_{C_t, N_t, K_{t+1}} \quad E\left[\sum_{t=0}^{\infty} \beta^t U(C_t, N_t)\right],$$

subject to (34) and (A.14). We can derive the first-order conditions for this economy:

$$\frac{-U_2(C_t, N_t)}{U_1(C_t, N_t)} = A_t F_2(K_t, A_t N_t), \tag{A.20}$$

$$G'(X_t/K_t) Q_t = 1, \tag{A.21}$$

$$\begin{aligned} Q_t \quad = \quad & E_t \left\{ \frac{\beta U_1(C_{t+1}, N_{t+1})}{U_1(C_t, N_t)} [F_1(K_{t+1}, A_{t+1}N_{t+1}) + (1-\delta) Q_{t+1} \right. \\ & \left. + Q_{t+1} G(X_{t+1}/K_{t+1}) - Q_{t+1} G'(X_{t+1}/K_{t+1}) X_{t+1}/K_{t+1}] \right\}. \end{aligned} \tag{A.22}$$

where Q_t is the shadow price of capital. We can see that (A.21) is equivalent to (A.19). Using equation (A.21), we can see that (A.22) is equivalent to (A.16). This completes the proof of the isomorphic result. Q.E.D.

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