Sectoral Bubbles and Endogenous Growth*

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Abstract

Stock price bubbles are often on productive assets and occur in a sector of the economy. In addition, their occurence is often accompanied with credit booms. Incorporating these features, we provide a two-sector endogenous growth model with credit-driven stock price bubbles. Bubbles have a credit easing effect by relaxing collateral constraints and improving investment efficiency. Sectoral bubbles also have a capital reallocation effect in the sense that bubbles in a sector attract more capital to be allocated to that sector. Their impact on economic growth depends on the interplay between these two effects.

Keywords: Bubbles, Collateral Constraints, Externality, Economic Growth, Capital Reallocation

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1 Introduction

Financial crises are often accompanied with asset price bubbles and crashes. Major historical examples of asset-price bubbles include the Dutch Tulipmania in 1637, the South Sea bubble in England in 1720, the Mississippi bubble in France in 1720, the Roaring Twenties stock market bubble, the internet bubble in the late 1990s, the Japanese asset price bubble in the 1980s, China's stock and property bubble until 2007, and the US housing bubble until 2007. What causes asset price bubbles? What is the impact of such bubbles on the economy? How should policymakers respond to bubbles? While these general questions are central to macroeconomics, this paper aims to study the following specific question: How do bubbles affect long-run economic growth?

To address this question, we focus on a particular type of bubbles, the credit-driven bubbles, that have three important features.[1] First, bubbles are often accompanied with an expansion in credit following financial liberalization. The Japanese asset price bubble in the 1980s and China's stock market bubbles are examples. Another example is the recent US housing bubble. With this type of bubbles, the following chain of events is typical as described by Mishkin (2008): Optimistic beliefs about economic prospects raise the values of some assets. The rise in asset values encourages further lending against these assets and hence more investment in the assets. The rise in investment in turn raises asset values. This positive feedback loop can generate a bubble, and the bubble can cause the credit standards to ease as lenders become less concerned about the ability of borrowers to repay loans and instead rely on further rise of the asset values to shield themselves from losses.

Second, bubbles have real effects and affect market fundamentals. Take a stock price bubble as an example. The bubble in stock prices encourages more lending against the firms' assets and hence raises investment. The rise in investment raises capital accumulation and dividends.

Third, bubbles often appear in a particular sector or industry of the economy. For example, the China, Japan, and US housing bubbles all occurred in the real estate sector. The Roaring Twenties bubble and the internet bubble were based on speculation about the development of new technologies. The 1920s saw the widespread introduction of an amazing range of technological innovations including radio, automobiles, aviation and the deployment of electric power

[1] Mishkin (2008, 2010) argues that this type of bubbles is highly dangerous to the economy. There is a second type of bubbles that is less dangerous, which can be referred to as an irrational exuberance bubble. This type of bubbles may be driven by bounded rationality or behavioral biases. The Dutch Tulipmania is an example. Xiong and Yu (2011) show that the Chinese warrants bubble in 2005-2008 is another example.

grid. The 1990s was the decade when internet and e-commerce technologies emerged.

Incorporating the above features, we build a two-sector endogenous growth model with credit-driven stock price bubbles. We assume that the capital goods produced in one of the two sectors has a positive externality effect on the productivity of workers. This externality effect provides the growth engine of the economy, similar to that discussed by Arrow (1962), Sheshinski (1967), and Romer (1986). Unlike these models, we assume that financial markets are imperfect. In particular, firms in the two sectors face credit constraints in a way similar to that in Albuquerque and Hopenhayn (2004), Kiyotaki and Moore (1997) and Jermann and Quadrini (2011). In order to borrow from lenders, firms must pledge a fraction of their assets as collateral. In the event of default, lenders capture the collateralized assets and operate the firm with these assets. The loan repayment cannot exceed the stock market value of the firm with these assets. Otherwise, firms may take loans and walk away. The lenders then lose the loan repayment, but recover the smaller market value of the collateralized assets. When the collateral constraint is sufficiently tight, firms have an incentive to inflate their asset values so as to relax the collateral constraints. We call this effect of bubbles the credit easing effect. If lenders have optimistic beliefs about asset values and lend more to the firms, then firms can make more investment and raise their asset values. This positive feedback loop can support a bubble.

The credit easing effect of bubbles encourages investment and saving and hence enhances economic growth. In our two-sector model economy, bubbles have an additional capital reallocation effect: Bubbles in only one of the sectors attract more investment to that sector and may distort capital allocation between the two sectors. More specifically, if bubbles occur only in the sector that has positive externality, then bubbles partly correct the externality inefficiency and still enhance economic growth.

On the other hand, if bubbles occur only in the sector with no externality, then more capital is attracted to the sector that does not induce growth. The strength of this negative effect depends on the elasticity of substitution between the two types of capital goods produced in the two sectors. When the elasticity is large, the negative capital reallocation effect dominates the positive credit easing effect and hence bubbles retard growth. But when the elasticity is small, then an opposite result holds.

Our paper is closely related to the literature on the impact of bubbles on endogenous economic growth. Important studies include Saint-Paul (1992), Grossman and Yanagawa (1993), King and Ferguson (1993), Olivier (2000), and Hirano and Yanagawa (2010). The first three studies extend the overlapping generations model of Samuelson (1958), Diamond (1965), and Tirole (1985) to economies with endogenous growth due to externalities in capital accumulation. In their models, bubbles crowd out investment and reduce the growth rate of the economy. Using a similar model, Olivier (2000) shows that their results depend crucially on the assumption that bubbles are on unproductive assets. If bubbles are tied to R&D firms, then bubbles may enhance economic growth.

Unlike the preceding studies, Hirano and Yanagawa (2010) study bubbles in an infinite-horizon endogenous growth AK model with financial frictions. In their model, bubbles are on intrinsically useless assets,[2] and can be used to relax collateral constraints. They introduce investment heterogeneity and show that when the degree of pledgeability is relatively low, bubbles enhance growth. But when the degree of pledgeability is relatively high, bubbles retard growth.

Our paper is also related to the literature on bubbles in production economies with exogenous growth.[3] Closely related studies include Caballero, Farhi, and Hammour (2006), Caballero and Krishnamurthy (2006), Kocherlakota (2009), Farhi and Tirole (2010), Wang and Wen (2010), Martin and Ventura (2011a, 2011b), and Miao and Wang (2011). Caballero, Farhi, and Hammour (2006) introduce capital adjustment costs to the Diamond-Tirole model and study the episodes of speculative growth. Other studies focus on the effects of bubbles in the presence of financial frictions. Caballero and Krishnamurthy (2006) and Farhi and Tirole (2010) show that bubbles can provide liquidity and crowd investment in. Kocherlakota (2009), Martin and Ventura (2011a), and Miao and Wang (2011) show that bubbles can relax collateral constraints and improve investment efficiency. Miao and Wang (2011) provide a theory of credit-driven bubble that has the first two features discussed earlier.

Our paper builds on Miao and Wang (2011) and differs from previous studies in two major respects. First, bubbles in our model are attached to productive assets, rather than on intrinsically useless assets or assets with exogenous dividends. Second, our model economy features two sectors. Bubbles may occur in only one of the two sectors and attract too much capital to be allocated to that sector. Thus, sectoral bubbles have a capital reallocation effect, which may be detrimental to economic growth.

The remainder of the paper proceeds as follows. Section 2 sets up the model. Section 3

[2] This type of assets can be interpreted as money. The existence of bubbles explains why money has value. See Kiyotaki and Moore (2008) for a related model.

[3] See Kocherlakota (1992, 2008), Santos and Woodford (1997), and Hellwig and Lorenzoni (2009) for models of bubbles in pure exchange economies. See Brunnermeier (2009) for a survey of the literature on bubbles.

provides equilibrium characterizations. Section 4 studies the symmetric bubbly equilibrium in which bubbles occur in both sectors of the economy. Section 5 studies the asymmetric bubbly equilibrium in which bubbles occur in only one of the two sectors. Section 6 concludes. Technical proofs are collected in an appendix.

2 The Model

We consider a two-sector economy which consists of households, final goods producers, capital goods producers, and financial intermediaries. Time is continuous and the horizon is infinite. There is no aggregate uncertainty.

2.1 Households

There is a continuum of identical households with a unit mass. Each household derives utility from a consumption stream $\{C_t\}$ according to the following function:

$$\int_0^\infty e^{-\rho t} \log (C_t)\, dt,$$

where $\rho > 0$ is the subjective rate of time preference. Households supply labor inelastically. The labor supply is normalized to one. Households earn labor income, trade firm stocks, and make deposits to financial intermediaries (or banks). Financial intermediaries use deposits to make loans and earn zero profits. The net supply of any stock is normalized to one. Let r_t denote the interest rate. Because there is no aggregate uncertainty, the interest rate is equal to the rate of return on each stock. From the households' optimization problem, we can immediately derive the following first-order condition:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho. \tag{1}$$

2.2 Final Goods Producers

There is a continuum of identical final goods producers with a unit mass. Each final goods producer hires labor and rents two types of capital goods to produce output according to the following production function:

$$Y_t = A\omega^{\alpha-1}\left[\omega^{\frac{1}{\sigma}} k_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}} k_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}} (K_{1t}N_t)^{1-\alpha} \tag{2}$$

where k_{it} denotes the stock of type $i = 1, 2$ capital goods rented by a final goods producer, N_t denotes hired labor, K_{it} is the aggregate stock of type i capital, $\alpha \in (0, 1)$ represents the capital share, A represents total factor productivity, $\sigma > 0$ represents the elasticity of substitution between the two types of capital, and $\omega \in (0, 1)$ is a share parameter.

According to the specification of the production function in (2), type 1 capital goods have positive externality to the productivity of workers in individual firms, in the manner suggested by Arrow (1962), Sheshinski (1967) and Romer (1986). Unlike these studies, we differentiate between the two types of capital goods and assume that only one of them has positive externality. Intuitively, knowledge has a positive spillover effect. Knowledge is created and transmitted through human capital. Compared to human capital, it is more reasonable to assume that physical capital has no externality to the productivity of workers. We may view sector 1 as the sector producing human capital such as the education sector and view sector 2 as the manufacturing sector.

We adopt a functional form with constant elasticity of substitution between the two types of capital. When the elasticity $\sigma \to 1$, the production function approaches the Cobb-Douglas form. We will show later that the substitutability between the two types of capital has important implications for the impact of bubbles in the two sectors on economic growth.

Final goods producers behave competitively. Each final goods producer solves the following problem:

$$\max_{k_{1t}, k_{2t}, N_t} A\omega^{\alpha-1}\left[\omega^{\frac{1}{\sigma}} k_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}} k_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}} (K_{1t}N_t)^{1-\alpha} - w_t N_t - R_{1t}k_{1t} - R_{2t}k_{2t}, \tag{3}$$

where w_t denotes the wage rate, and R_{it} denotes the rental rate of type i capital, $i = 1, 2$. The first-order conditions are given by:

$$(1-\alpha)\frac{Y_t}{N_t} = w_t, \tag{4}$$

$$A\alpha\omega^{\frac{1}{\sigma}}\omega^{\alpha-1}(K_{1t}N_t)^{1-\alpha}\left[\omega^{\frac{1}{\sigma}} k_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}} k_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}-1} k_{1t}^{-\frac{1}{\sigma}} = R_{1t}, \tag{5}$$

and

$$A\alpha(1-\omega)^{\frac{1}{\sigma}}\omega^{\alpha-1}(K_{1t}N_t)^{1-\alpha}\left[\omega^{\frac{1}{\sigma}} k_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}} k_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}-1} k_{2t}^{-\frac{1}{\sigma}} = R_{2t}. \tag{6}$$

When solving the optimization problem, individual firms take the factor prices and aggregate capital stock K_{1t} in sector 1 as given.

Because there is a unit mass of identical final goods producers, the aggregate capital stock is equal to a representative firm's capital stock in that $k_{it} = K_{it}$. In addition, Y_t represents aggregate output.

2.3 Capital Goods Producers

The two types of capital goods are produced in two sectors, respectively. Each sector has a continuum of ex ante identical capital goods producers with a unit mass. They are heterogeneous ex post because they face idiosyncratic investment opportunities. As in Kiyotaki and Moore (1997, 2005, 2008), each firm meets an investment opportunity with probability πdt from time t to $t + dt$. With probability $1 - \pi dt$, no investment opportunity arrives. This assumption captures firm-level investment lumpiness and generates ex post firm heterogeneity. As will be shown below, it is also useful for Tobin's marginal Q to be greater than 1 in equilibrium. Assume that the arrival of investment opportunities is independent across firms and over time.

We write the law of motion for capital of firm j in sector i between time t and $t + dt$ as:

$$K_{it+dt}^{j} = \begin{cases} \left(1 - \delta dt\right) K_{it}^{j} + I_{it}^{j} & \text{with probability } \pi dt \\ \left(1 - \delta dt\right) K_{it}^{j} & \text{with probability } 1 - \pi dt \end{cases}, \tag{7}$$

where $\delta > 0$ is the depreciation rate of capital and I_{it}^{j} is the investment level.

Each firm's objective is to maximize its stock market value. Let $V_{it}(K_{it}^{j})$ be the value function, which represents the stock market value of firm j in sector i when its capital stock is K_{it}^{j}. Then it satisfies the asset pricing equation:

$$V_{i0}\left(K_{i0}^{j}\right) = \max_{I_{it}^{j}} \int_{0}^{T} e^{-\int_0^t r_s ds}\left(R_{it}K_{it}^{j} - \pi I_{it}^{j}\right) dt + e^{-\int_0^T r_s ds} V_{iT}\left(K_{iT}^{j}\right), \text{ any } T > 0, \tag{8}$$

subject to the law of motion (7) and two additional constraints. These two constraints reflect financial frictions. The first constraint is given by:

$$I_{it}^{j} \leq R_{it}K_{it}^{j} + L_{it}^{j}, \tag{9}$$

where L_{it}^{j} represents bank loans. This constraint states that firms use internal funds $R_{it}K_{it}^{j}$ and bank loans L_{it}^{j} to finance investment. We assume that firms do not raise new equity. This assumption reflects the fact that equity finance is more costly than debt finance. For analytical tractability, we consider loans without interest payments as in Jermann and Quadrini (2010). Incorporating loans with interests would make loan volume become a state variable, which complicates the analysis of a firm's optimization problem. See Miao and Wang (2011) for an analysis of the model with one-period debt with interests in a discrete time framework.

The second constraint is the collateral constraint given by:

$$L_{it}^{j} \leq V_{it}\left(\xi K_{it}^{j}\right), \tag{10}$$

where $\xi \in (0,1)$. For simplicity, we assume that all firms in the economy face the same degree of pledgeability, represented by the parameter ξ.[4] This parameter represents the degree of financial frictions. The motivation for this collateral constraint is similar to that in Kiyotaki and Moore (1997). In order to borrow from a bank, firm j must pledge a fraction ξ of its assets as collateral or effectively pledge the stock market value of the firm with assets ξK_{it}^j as collateral.[5] The bank never allows the loan repayment L_{it}^j to exceed the stock market value $V_{it}(\xi K_{it}^j)$ of the pledged assets. If this condition is violated, then firm j may take loans L_{it}^j and walk away, leaving the collateralized asset ξK_{it}^j behind. In this case, the bank operates the firm with the collateralized assets ξK_{it}^j and obtains the smaller firm value $V_{it}(\xi K_{it}^j)$, which is the collateral value.

The collateral constraint in (10) may be interpreted as an incentive constraint in an optimal contract when the borrowers have limited commitment:[6] Firm j may default on debt. If it happens, then the firm and the bank renegotiate the loan repayment. In addition, the bank reorganizes the firm. Because of default costs, the bank can only seize a fraction ξ of existing capital K_{it}^j. Alternatively, we may interpret ξ as an efficiency parameter in that the bank may not be able to efficiently use the firm's assets K_{it}^j. The bank can run the firm with these assets and obtains firm value $V_{it}(\xi K_{it}^j)$. Or it can sell these assets to a third party at the *going-concern value* $V_{it}(\xi K_{it}^j)$ if the third party can run the firm using assets ξK_{it}^j. This value is the threat value to the bank. Following Jermann and Quadrini (2010), we assume that the firm has all the bargaining power in the renegotiation and the bank gets only the threat value. The key difference between our modeling and theirs is that the threat value to the bank is the going concern value in our model, while Jermann and Quadrini (2010) assume that the bank liquidates the firm's assets and obtains the liquidation value in the even of default.[7]

Enforcement requires that the value (to the firm) of not defaulting is not smaller than the value of defaulting, that is,

$$V_{it}(K_{it}^j) - L_{it}^j \geq V_{it}(K_{it}^j) - V_{it}(\xi K_{it}^j). \tag{11}$$

[4] As will be analyzed below, this assumption also allows us to isolate the distortional effect on capital allocation across the two sectors caused by sectoral bubbles from that caused by different degrees of pledgeability.

[5] Alternatively, we may assume that the firm pledges a fraction ξ of the stock market value of the firm, $V_{it}\left(K_{it}^j\right)$, as collateral. The collateral value is $\xi V_{it}\left(K_{it}^j\right)$. This modeling does not change our key insights. See Martin and Ventura (2011a,b) for related credit constraints.

[6] See Albuquerque and Hopenhayn (2004) and Alvarez and Jermann (2000) for related contracting problems.

[7] U.S. Bankruptcy law has recognized the need to preserve going concern value when reorganizing businesses in order to maximize recoveries by creditors and shareholders (see 11 U.S.C. §1101 et seq.). Bankruptcy laws seek to preserve going concern value whenever possible by promoting the reorganization, rather than the liquidation, of businesses.

This incentive constraint is equivalent to the collateral constraint in (10).

Note that the modeling of the collateral constraint in (10) follows from Miao and Wang (2011) who also provide a detailed discussion of the optimal contract. It is different from that in Kiyotaki and Moore (1997):

$$L_{it}^{j} \leq \xi Q_{it} K_{it}^{j}, \tag{12}$$

where Q_{it} represents the shadow price of capital produced in sector i. The expression $\xi Q_{it} K_{it}^{j}$ is the shadow value of the collateralized assets or the liquidation value.[8] This form of collateral constraint rules out bubbles. By contrast, according to (10), we allow the collateralized assets are valued in the stock market as the going-concern value when the new owner can use these assets to run the reorganized firm after default. If both firms and lenders believe that firms' assets may be overvalued due to stock market bubbles, then these bubbles will relax the collateral constraint, which provides a positive feedback loop mechanism..

2.4 Competitive Equilibrium

Let $I_{it} = \int I_{it}^{j} dj$ and $K_{it} = \int K_{it}^{j} dj$ denote aggregate investment and aggregate capital in sector i. A competitive equilibrium consists of trajectories (C_t), (K_{it}), (I_{it}), (Y_t), (r_t), (w_t), and (R_{it}), $i = 1, 2$, such that:

(i) Households optimize so that equation (1) holds.

(ii) Each firm j solves problem (8) subject to (7), (9) and (10).

(ii) Rental rates satisfy:

$$R_{1t} = A\alpha\omega^{\frac{1}{\sigma}}\omega^{\alpha-1}K_{1t}^{1-\alpha}\left[\omega^{\frac{1}{\sigma}}K_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}}K_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}-1} K_{1t}^{-\frac{1}{\sigma}}, \tag{13}$$

and

$$R_{2t} = A\alpha(1-\omega)^{\frac{1}{\sigma}}\omega^{\alpha-1}K_{1t}^{1-\alpha}\left[\omega^{\frac{1}{\sigma}}K_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}}K_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}-1} K_{2t}^{-\frac{1}{\sigma}}. \tag{14}$$

(iii) The wage rate satisfies (4) for $N_t = 1$.

(iv) Markets clear in that:

$$C_t + \pi\left(I_{1t} + I_{2t}\right) = Y_t = A\omega^{\alpha-1}K_{1t}^{1-\alpha}\left[\omega^{\frac{1}{\sigma}}K_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}}K_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}}. \tag{15}$$

[8] Note that our model differs from the Kiyotaki and Moore model in market arrangements, besides other specific modeling details. Kiyotaki and Moore assume that there is a market for physical capital, but there is no stock market for trading firm shares. In addition, they assume that households and entrepreneurs own firms and trade physical capital in the capital market. By contrast, we assume that households trade firm shares in the stock market and that firms own physical captial and make investment

To write equations (13), (14), and (15), we have imposed the market-clearing conditions $k_{it} = K_{it}$ and $N_t = 1$ in equations (5), (6), and (2).

3 Equilibrium Characterization

In this section, we first analyze a single firm's decision problem. We then conduct aggregation and characterize equilibrium by a system of differential equations. Finally, we study the balanced growth path in the bubbleless equilibrium.

3.1 A Single Firm's Decision Problem

We take the interest rate r_t and rental rates R_{1t} and R_{2t} as given and study a capital goods producer's decision problem (8) subject to (9) and (10). We conjecture that the value function takes the following form:

$$V_{it}(K_{it}^j) = Q_{it}K_{it}^j + B_{it}, \tag{16}$$

where Q_{it} and B_{it} are to be determined variables. We interpret Q_{it} as the shadow price of capital, or marginal Q following Hayashi (1982). We will show below that both $B_{it} = 0$ and $B_{it} > 0$ may be part of the equilibrium solution because the the firm's dynamic programming problem does not give a contraction mapping. We interpret $B_{it} > 0$ as the bubble component of the asset value. We will refer to the equilibrium with $B_{it} = 0$ for all t as the bubbleless equilibrium and to the equilibrium with $B_{it} > 0$ as the bubbly equilibrium.

When $B_{it} = 0$, marginal Q is equal to average Q, $V_{it}(K_{it}^j)/K_{it}^j$, a result similar to that in Hayashi (1982). In this case, the collateral constraint (10) becomes (12), a form used in Kiyotaki and Moore (1997). When $B_{it} > 0$, the collateral constraint becomes:

$$L_{it}^j \leq V_{it}(\xi K_{it}^j) = \xi Q_{it}K_{it}^j + B_{it}. \tag{17}$$

Thus, firm j can use the bubble B_{it} to raise the collateral value and relax the collateral constraint. In this way, firm j can make more investment and raise the market value of its assets. We call this effect of bubbles the credit easing effect. If lenders believe that firm j's assets have high a value possibly because of the existence of bubbles and if lenders decide to lend more to firm j, then firm j can borrow more and invest more, thereby making its assets indeed more valuable. This process is self-fulfilling and a bubble may sustain.

The following proposition characterizes the solution to a firm's optimization problem.

Proposition 1 *Suppose $Q_{it} > 1$. Then (i) the market value of the firm is given by (16); (ii) optimal investment is given by*

$$I_{it}^{j} = \left(R_{it} + \xi Q_{it}\right) K_{it}^{j} + B_{it}, \tag{18}$$

and (iii) (B_{it}, Q_{it}) satisfy the following differential equations:

$$r_t Q_{it} = R_{it} + \left(R_{it} + \xi Q_{it}\right) \pi \left(Q_{it} - 1\right) - \delta Q_{it} + \dot{Q}_{it}, \tag{19}$$

$$r_t B_{it} = \pi \left(Q_{it} - 1\right) B_{it} + \dot{B}_{it}, \tag{20}$$

and the transversality condition:

$$\lim_{T \to \infty} \exp\left(-\int_0^T r_s ds\right) Q_{iT} K_{iT}^{j} = 0, \quad \lim_{T \to \infty} \exp\left(-\int_0^T r_s ds\right) B_{iT} = 0. \tag{21}$$

Investment decisions are described by the Q theory (Tobin (1969) and Hayashi (1982)). In the absence of adjustment costs, when $Q_{it} > 1$, firms make investment and the optimal investment level reaches the upper bound given in (9). In addition, the collateral constraint in (10) or (17) is binding. We then obtain equation (18). Equation (19) is an asset pricing equation for capital. The expression on the left hand side represents the return on capital and the expressions on the right hand side represent dividends plus capital gains. Dividends are equal to the rental rate or the marginal product of capital R_{it} plus the return from new investment $\left(R_{it} + \xi Q_{it}\right) \pi \left(Q_{it} - 1\right)$ minus the depreciated value δQ_{it}. An additional unit of capital generates $R_{it} + \xi Q_{it}$ units of new investment, when an investment opportunity arrives. Each unit of new investment raises firm value by $\pi \left(Q_{it} - 1\right)$ on average.

Equation (20) is an asset pricing equation for the bubble $B_{it} > 0$. We may rewrite it as

$$\frac{\dot{B}_{it}}{B_{it}} + \pi(Q_{it} - 1) = r_t, \text{ for } B_t > 0. \tag{22}$$

It states that the rate of return on the bubble r_t is equal to the rate of capital gains $\dot{B}_{it}/B_{it}$ plus dividend yields $\pi \left(Q_{it} - 1\right)$. The dividend yields come from the fact that a dollar of the bubble allows the firm to make one more dollar of investment and raises firm value by $\left(Q_{it} - 1\right)$. Because investment opportunities arrive at the rate π, the average benefit is equal to $\pi \left(Q_{it} - 1\right)$. Most models in the literature study bubbles on intrinsically useless assets. In this case, the return on the bubble is equal to the capital gain. Thus, the growth rate of the bubble is equal to the interest rate. As a result, the transversality condition (21) will rule out bubbles. In our model, bubbles are on productive assets and their growth rate is less than the interest rate.

Thus, they cannot be ruled out by the transversality condition. As Santos and Woodford (1997) point out, it is very hard to generate bubbles in an infinite horizon economy. It is possible to generate bubbles in overlapping-generations models when the economy is dynamically inefficient (see Tirole (1985)).

3.2 Equilibrium System

We can use the decision rule described in Proposition 1 to easily conduct aggregation and derive equilibrium conditions.

Proposition 2 *Suppose $Q_{it} > 1$. Then the equilibrium dynamics for $(B_{it}, Q_{it}, K_{it}, I_{it}, C_t, Y_t)$ satisfy the following system of differential equations:*

$$\dot{K}_{it} = -\delta K_{it} + \pi I_{it}, \quad K_{i0} \text{ given,} \tag{23}$$

$$I_{it} = (R_{it} + \xi Q_{it}) K_{it} + B_{it}, \tag{24}$$

together with (15), (19)-(20), and the transversality condition:

$$\lim_{T\to\infty} \exp\left(-\int_0^T r_s ds\right) Q_{iT} K_{iT} = 0, \quad \lim_{T\to\infty} \exp\left(-\int_0^T r_s ds\right) B_{iT} = 0, \tag{25}$$

where R_{1t} and R_{2t} satisfy (13) and (14), respectively, and r_t satisfies (1).

We shall focus on the long-run steady-state equilibrium in which a long-run balanced growth path exists. We will not study transitional dynamics. In a balanced growth path, all variables grow at possibly different constant rates. In particular, the growth rates of some variables may be zero.

The condition $Q_{it} > 1$ enables us to apply Proposition 1. This condition is generally hard to verify because Q_{it} is an endogenous variable. We will show below that Q_{it} is constant along the balanced growth path. We shall impose assumptions on the primitive parameters such that $Q_{it} > 1$ on the balanced growth path.

3.3 Bubbleless Equilibrium

We start by analyzing the bubbleless equilibrium in which $B_{it} = 0$ for all t. On a balanced growth path, consumption grows at the constant rate. By the resource constraint (15), aggregate capital, aggregate investment, and output all grow at the same rate. By equation (1), the interest rate r_t must be constant.

To determine the endogenous growth rate, we need to derive the investment rule. As we show in Proposition 1, if $Q_{it} > 1$, then both the investment constraint (9) and the collateral constraint (12) will bind. Intuitively, this case will happen when the collateral constraint is sufficiently tight or ξ is sufficiently small. When ξ is sufficiently large, then firms will have enough funds to finance investment and the collateral constraint will not bind. In this case, firms effectively do not face financial frictions and $Q_{it} = 1$.

Specifically, in the case without financial frictions, we can show that

$$R_{1t} = R_{2t} = R^* \equiv \alpha A, \tag{26}$$

and

$$\frac{\omega}{1-\omega} = \frac{K_{1t}}{K_{2t}}. \tag{27}$$

Defining $K_t = K_{1t} + K_{2t}$, we then obtain

$$K_{1t} = \omega K_t,\ K_{2t} = (1-\omega)K_t. \tag{28}$$

on the balanced growth path. Equation (27) or (28) gives the capital allocation rule across the two sectors under perfect financial markets.[9] Using equation (28), we can also derive aggregate output on the balanced growth path:

$$Y_t = A\omega^{\alpha-1}K_{1t}^{1-\alpha}\left[\omega^{\frac{1}{\sigma}}K_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}}K_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}} = AK_t. \tag{29}$$

Because aggregate output is linear in the aggregate capital stock, our two-sector endogenous growth model without financial frictions is isomorphic to a one-sector AK model. We denote the economic growth rate by g_0. Because of externality in the decentralized economy, this growth rate is still less than that in an economy in which a social planner makes the consumption and investment decisions.

We denote the economic growth rate by g^* for the case of binding collateral constraints. By equations (23) and (24), we obtain:

$$g^* = \frac{\dot{K}_{it}}{K_{it}} = -\delta + \pi\left(R_{it} + \xi Q_{it}\right), \tag{30}$$

if $Q_{it} > 1$. Thus, $R_{it}+\xi Q_{it}$ must be constant. In the appendix, we will show that $R_{1t} = R_{2t} = R^*$ and $Q_{1t} = Q_{2t} = Q^*$ for some constant Q^*. By equations (13) and (14), equations (27) and

[9] Note that this allocation rule is not socially efficient because private firms do not internalize the externality effect from sector 1.

(28) still hold. In addition, equation (29) also holds. Thus, collateral constraints do not distort capital allocation between the two sectors. The reason is that we have assumed that the two sectors face identical collateral constraints (i.e., identical ξ). If the pledgeability parameter ξ were different across the two sectors, then the capital allocation between the two sectors would be distorted due to financial frictions. Our model isolates this effect from the distortion caused by sectoral bubbles.

Next, we rewrite equation (19) on the balanced growth path:

$$(r+\delta)\,Q^{*}=R^{*}+\pi(R^{*}+\xi Q^{*})(Q^{*}-1). \tag{31}$$

Substituting $r=g^{*}+\rho$ using (1), $R^{*}=\alpha A$, and equation (30) into equation (31), we can solve for Q^{*} :

$$Q^{*}=\frac{(1-\pi)\alpha A}{\rho+\pi\xi}. \tag{32}$$

and the long-run growth rate g^{*}. We summarize the above analysis in the following result:

Proposition 3 *Suppose*

$$\alpha A-\rho-\delta>0. \tag{33}$$

(i) If

$$\xi>\frac{A\alpha(1-\pi)}{\pi}-\frac{\rho}{\pi}, \tag{34}$$

then consumption, capital, and output on the balanced growth path grow at the rate

$$g_{0}=\alpha A-\rho-\delta. \tag{35}$$

(ii) If

$$\xi<\frac{A\alpha(1-\pi)}{\pi}-\frac{\rho}{\pi}, \tag{36}$$

$$\frac{\alpha A\pi\,(\rho+\xi)}{\rho+\pi\xi}>\delta, \tag{37}$$

then consumption, capital, and output on the balanced growth path grow at the rate

$$g^{*}=\frac{\alpha A\pi\,(\rho+\xi)}{\rho+\pi\xi}-\delta<g_{0}. \tag{38}$$

Condition (33) is a technical condition that ensures $g_{0}>0$. Condition (34) says that if capital goods producers can pledge sufficient assets as collateral or ξ is sufficiently large, then the collateral constraints are so loose that they are never binding. In this case, capital goods producers can achieve investment efficiency in that $Q_{it}=1$ for $i=1,2$. However, final goods producers cannot achieve investment efficiency because they do not internalize the externality from the aggregate capital stock in sector 1. We then obtain the familiar growth rate g_0 as in the standard AK model of learning by doing without financial frictions. This rate is smaller than the first-best socially optimal growth rate, $(A - \rho - \delta)$.

Condition (36) ensures that $Q^* > 1$ so that we can apply Propositions 1-2. From conditions (34) and (36), we observe that the arrival rate π must be sufficiently small for $Q^* > 1$ and hence financial frictions matter. Condition (37) is a technical condition that ensures $g^* > 0$. These two conditions are equivalent to

$$\frac{\rho\left(\delta - \alpha A \pi\right)}{\pi\left(\alpha A - \delta\right)} < \xi < \frac{A\alpha(1-\pi)}{\pi} - \frac{\rho}{\pi}.$$

One can show that condition (33) makes the two inequalities possible.

To understand the intuition behind the determinant of growth, we add up equations in (23) for $i = 1, 2$ and notice that on the balanced growth path aggregate capital grows at a constant rate g. We then obtain

$$g = -\delta + \frac{\pi\left(I_{1t} + I_{2t}\right)}{K_{1t} + K_{2t}} = -\delta + s\frac{Y_t}{K_t}, \tag{39}$$

where $s = \pi\left(I_{1t} + I_{2t}\right)/Y_t$ is the aggregate investment rate or the aggregate saving rate. Both the aggregate saving rate and the output-capital ratio are constant along a balanced growth path. They are the key determinants of long-run growth.

In the bubbleless equilibrium, we have shown that $Y_t = AK_t$ so that the output-capital ratio is equal to A. By equation (38), the aggregate saving rate s is equal to $\alpha\pi\left(\rho + \xi\right)/\left(\rho + \pi\xi\right)$. Now we can understand that the growth rate g^* in the bubbleless equilibrium depends on the parameters A, α, ρ and δ and the impact of these parameters on g^* is qualitatively identical to that in the standard AK models of learning by doing (e.g., Romer (1986)). In our model with collateral constraint and investment frictions, two new parameters π and ξ also affect the growth rate g^*. We can easily show that g^* increases with π. Intuitively, the economy will grow faster if more firms have investment opportunities or if individual firms meet investment opportunities more frequently. We can also show that g^* increases with ξ. The intuition is that an increase in ξ relaxes the collateral constraints, thereby enhancing investment efficiency and raising the investment rate. The parameter ξ may proxy for the extent of financial development. An implication of Proposition 3 is that economies with more developed financial markets grow faster.

4 Symmetric Bubbly Equilibrium

In this section, we study symmetric bubbly equilibrium in which $B_{it} > 0$ for some t for $i = 1, 2$. Let consumption C_t grow at the constant rate g_b on the balanced growth path. By (1), the interest rate r_t is constant on the balanced growth path and is equal to

$$r = g_b + \rho. \tag{40}$$

In addition, by equations (15), (23), and (24), K_{it}, I_{it}, Y_t, and B_{it} all grow at the same rate g_b on the balance growth. In this case, equation (20) becomes:

$$r = g_b + \pi(Q_{it} - 1). \tag{41}$$

Thus, on the balance growth path, the capital price Q_{it} is constant for $i = 1, 2$. We denote this constant by Q_b. It follows from the above two equations that

$$Q_{1t} = Q_{2t} = Q_b = \frac{r - g_b}{\pi} + 1 = \frac{\rho}{\pi} + 1. \tag{42}$$

This equation shows that $Q_b > 1$ so that we can apply Propositions 1-2 on the balanced growth path. On the balanced growth path, equation (19) becomes:

$$(r + \delta)\, Q_b = R_{it} + \pi(R_{it} + \xi Q_b)(Q_b - 1). \tag{43}$$

Thus, R_{1t} and R_{2t} are equal to the same constant, denoted by R_b.

As in Section 3.3, we can show that the allocation rule under perfect financial markets given in (28) holds on the balanced growth path. Consequently, the rental rates are given by:

$$R_{1t} = R_{2t} = R_b = \alpha A, \tag{44}$$

and aggregate output is given by $Y_t = AK_t$.

The above analysis demonstrates that the presence of bubbles in both sectors does not distort capital allocation across the two sectors. This result depends on the fact that the two sectors face the same degree of financial frictions as described by the identical parameter ξ. If the two sectors faced different values of ξ, then it follows from equation (43) that the factor prices R_{1t} and R_{2t} in the two sectors would be different. As a result, capital allocation across the two sectors will be distorted in that equation (28) will not hold.

Isolating the capital allocation effect of bubbles, we find that the role of bubbles is to relax the collateral constraints and to improve investment efficiency. In addition, equations (23) and

(24) imply that on the balanced growth path,

$$g_b = \frac{\dot{K}_{it}}{K_{it}} = -\delta + \pi \left(R_b + \xi Q_b + \frac{B_{it}}{K_{it}} \right). \tag{45}$$

Thus, the presence of bubbles $B_{it}/K_{it} > 0$ enhances economic growth.

Proposition 4 *Suppose condition (37) and the following condition hold:*

$$\xi < \frac{\alpha A(1-\pi)}{\rho+\pi} - \frac{\rho}{\pi}. \tag{46}$$

Then, on the balanced growth path, (i) both the bubbleless equilibrium and the symmetric bubbly equilibrium exist; (ii) the economic growth rate in the symmetric bubbly equilibrium is given by:

$$g_b = \frac{\alpha A\pi(1+\rho)}{\rho+\pi} + \rho\xi - \rho - \delta; \tag{47}$$

and (iii) $g^* < g_b < g_0$.

Condition (46) ensures that bubbles are positive, $B_{it}/K_{it} > 0$. Note that this condition implies condition (36) also holds. Under the additional condition (37), we deduce that the steady-state bubbleless equilibrium also exists. We can also show that $g_b > g^* > 0$. The intuition behind this result is as follows. Since bubbles in the two sectors relax the collateral constraints and raise the aggregate investment rate or the saving rate, the growth rate in the symmetric bubbly equilibrium is higher than that in the bubbleless equilibrium. However, it is still smaller than the growth rate in the economy without the collateral constraints. The reason is that the collateral constraints in the presence of bubbles are not sufficiently loose. They are still binding and cause investment inefficiency.

5 Asymmetric bubbly Equilibrium

In this section, we study asymmetric bubbly equilibrium in which bubbles appear in only one of the two sectors. Recall that only capital goods produced in sector 1 have positive externality to produce final output. Because capital goods produced in the two sectors have a different role in the economy, bubbles in one of the two sectors may have different impact on economic growth than bubbles in the other sector.

5.1 Bubble in the Sector with Externality

We first consider asymmetric bubbly equilibrium in which $B_{1t} > 0$ and $B_{2t} = 0$ for all t. On the balanced growth path, consumption, capital, investment, output, and bubbles should grow at the same rate. Denote this rate by g_{1b}. By equations (1) and (20), we obtain:

$$r = g_{1b} + \rho, \tag{48}$$

$$r = g_{1b} + \pi(Q_1 - 1). \tag{49}$$

Thus, the interest rate r_t and the capital price Q_{1t} in sector 1 are constants, denoted by r and Q_1 respectively. The above two equations imply that:

$$Q_1 = \frac{\rho}{\pi} + 1 > 1. \tag{50}$$

Using equation (19), we deduce that on the balanced growth path,

$$(r + \delta) Q_1 = R_{1t} + \pi(R_{1t} + \xi Q_1)(Q_1 - 1). \tag{51}$$

Thus, the rental rate R_{1t} for type 1 capital is equal to a constant, denoted by R_1. Substituting equation (48) and (50) into equation (51) yields:

$$R_1 = \frac{1}{\pi}\frac{\rho + \pi}{1 + \rho}\left[\rho(1 - \xi) + \delta + g_{1b}\right]. \tag{52}$$

Next, we derive the rental rate and the capital price in sector 2. We use equations (13)-(14) to show that:

$$\left(\frac{R_{1t}}{R_{2t}}\right)^{\sigma} = \frac{\omega}{(1 - \omega)}\frac{K_{2t}}{K_{1t}}. \tag{53}$$

Plugging this equation and $R_{1t} = R_1$ into equation (13), we obtain:

$$R_1 = A\alpha\omega^{\frac{1}{\sigma}+\alpha-1}\left[\omega^{\frac{1}{\sigma}} + \frac{1-\omega}{\omega^{\frac{\sigma-1}{\sigma}}}\left(\frac{R_1}{R_{2t}}\right)^{\sigma-1}\right]^{\frac{\alpha\sigma-\sigma+1}{\sigma-1}}. \tag{54}$$

Thus, R_{2t} must be equal to a constant, denoted by R_2. We will show below that R_1 is not equal to R_2 in the asymmetric bubbly equilibrium, unlike in the symmetric bubbly equilibrium. As a result, capital allocation across the two sectors is distorted. We call this effect of bubbles the capital reallocation effect. As revealed by equation (53), the strength of the capital reallocation effect depends crucially on the elasticity of substitution parameter σ.

On the balanced growth path, equations (23) and (24) imply that

$$g_{1b} = \frac{\dot{K}_{1t}}{K_{1t}} = -\delta + \pi\left(R_1 + \xi Q_1 + \frac{B_{1t}}{K_{1t}}\right), \tag{55}$$

$$g_{1b} = \frac{\dot{K}_{2t}}{K_{2t}} = -\delta + \pi(R_2 + \xi Q_{2t}). \tag{56}$$

Thus, Q_{2t} is also equal to a constant, denoted by Q_2. Using equation (19) and (48), we obtain:

$$(\rho + g_{1b} + \delta)\, Q_2 = R_2 + \pi(R_2 + \xi Q_2)(Q_2 - 1). \tag{57}$$

Combining equations (56)-(57) and eliminating g_{1b} yields:

$$Q_2 = \frac{1-\pi}{\pi\xi + \rho} R_2. \tag{58}$$

Substituting this equation into (56) yields:

$$R_2 = \frac{1}{\pi} \frac{\pi\xi + \rho}{\rho + \xi} (\delta + g_{1b}). \tag{59}$$

Substituting equations (52) and (59) into (54) yields a nonlinear equation for g_{1b}. We also need to solve for the bubble to capital ratio, $B_{1t}/K_{1t} > 0$, using equation (55). The following proposition summarizes the result.

Proposition 5 *Suppose that there exists a unique solution* (R_1, R_2, g_{1b}) *to the system of equations (52), (54) and (59).*[10] *Suppose that:*

$$g_{1b} > \frac{(\rho+\xi)(\rho+\pi)}{1-\pi} - \delta > 0. \tag{60}$$

Then the steady-state asymmetric bubbly equilibrium with $B_{1t} > 0$ *and* $B_{2t} = 0$ *exists and the economic growth rate is* g_{1b}.

We can use equations (58)-(59) and condition (60) to check that $Q_2 > 1$. Since $Q_1 > 1$ by (50), our use of Propositions 1-2 in deriving Proposition 5 is justified.

Condition (60) guarantees the existence of $B_{1t}/K_{1t} > 0$. Given this condition, we can use equations (52) and (59) to show that $R_1 < R_2$. Intuitively, the existence of bubbles in sector 1 relaxes the collateral constraints for firms in that sector, thereby attracting more investment in sector 1. As a result, capital moves more to sector 1 instead of sector 2, causing the factor price in sector 1 to be smaller than that in sector 2, i.e., $R_1 < R_2$.

[10] Since this system is highly nonlinear, we are unable to provide explicit existence conditions in terms of primitive parameter values. In Section 5.3, we provide some numerical examples to illustrate the existence.

5.2 Bubble in the Sector without Externality

Now, we consider the asymmetric bubbly equilibrium in which $B_{1t} = 0$ and $B_{2t} > 0$ for all t. We use g_{2b} to denote the common growth rate of consumption, capital, investment, output, and the bubble in sector 2. We can follow a similar analysis to that in the previous subsection to derive the following proposition. We omit its proof.

Proposition 6 *Suppose that there exists a unique solution (R_1, R_2, g_{2b}) to the following system of equations:*

$$R_1 = \frac{1}{\pi}\frac{\pi\xi + \rho}{\rho + \xi}(\delta + g_{2b}), \tag{61}$$

$$R_2 = \frac{1}{\pi}\frac{\rho + \pi}{1 + \rho}\left[\rho(1 - \xi) + \delta + g_{2b}\right], \tag{62}$$

together with (54). Suppose that

$$g_{2b} > \frac{(\rho + \xi)(\rho + \pi)}{1 - \pi} - \delta > 0. \tag{63}$$

Then the steady-state asymmetric bubbly equilibrium with $B_{1t} = 0$ and $B_{2t} > 0$ exists and the economic growth rate is g_{2b}.

Condition (63) ensures that $B_{2t}/K_{2t} > 0$. It also implies that $R_2 < R_1$. The intuition is that the existence of bubbles in sector 2 attracts more capital to move from sector 1 to sector 2.

5.3 Do Bubbles Enhance or Retard Growth?

In Proposition 4, we have shown that the presence of bubbles in the two sectors enhances long-run growth. The intuition is that bubbles relax collateral constraints and improve investment efficiency. We have called this effect the credit easing effect. In the last two subsections, we have shown that the presence of bubbles in only one of the two sectors has an additional capital reallocation effect: It causes capital allocation between the two sectors to be distorted, relative to that in a bubbleless equilibrium. Bubbles in one sector attract more investment to that sector, causing more accumulation of capital in that sector. Intuitively, if the capital stock in that sector has a positive spillover effect on the economy, bubbles in that sector will enhance growth. On the other hand, if bubbles appear only in the sector without a positive spillover effect, then they may retard growth. The preceding capital reallocation effect depends on the substitutability between the two types of capital goods. If the elasticity of substitution between the two types of capital goods is large, then the reallocation effect will be large. The following proposition formalizes the above intuition.

Proposition 7 *Suppose that the conditions in Propositions 3(ii) and 4-6 hold. (i) If* $\sigma > \frac{1}{1-\alpha}$, *then*

$$g_{2b} < g^* < g_b < g_{1b}.$$

(ii) If $0 < \sigma < \frac{1}{1-\alpha}$, *then*

$$g^* < g_{1b} < g_b \text{ and } g^* < g_{2b} < g_b.$$

(iii) If $\sigma = \frac{1}{1-\alpha}$, *then*

$$g_{2b} = g^* < g_b = g_{1b}.$$

To understand this proposition, we define $\beta = K_{1t}/K_t$ and use the expression for Y_t in equation (15) to derive the capital-output ratio as

$$\frac{Y_t}{K_t} = A\omega^{\alpha-1}\left[\omega^{\frac{1}{\sigma}}\beta^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}}(1-\beta)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma}{\sigma-1}}\beta^{1-\alpha}. \tag{64}$$

Plugging this equation into (39) reveals that the aggregate saving rate s and the share of type 1 capital goods β are important determinants of the economic growth rate. The impact of bubbles on the economic growth rate is through these two variables. In particular, the credit easing effect relaxes the collateral constraints and raises the aggregate saving rate s. The capital reallocation effect influences capital allocation between the two sectors represented by β and hence the output-capital ratio.

In both the bubbleless and the symmetric bubbly equilibria, we have shown that $\beta = \omega$. Thus, symmetric bubbles do not have a capital reallocation effect. As shown in Proposition 4, these bubbles raise the aggregate saving rate s and hence $g_b > g^*$.

Asymmetric bubbles have a capital reallocation effect, causing $\beta \neq \omega$. When bubbles appear in sector 1 only, we have shown in Section 5.1 that $\beta > \omega$. Since type 1 capital has a positive externality effect, more capital allocation to Sector 1 raises the output-capital ratio. Thus, the capital reallocation effect enhances economic growth. However, since only sector 1 has bubbles, the credit easing effect will be smaller than that for the case of symmetric bubbles. The capital reallocation effect can be strong enough to more than offset the weaker credit easing effect if the elasticity of substitution between the two types of capital goods is large enough. This explains why $g_b < g_{1b}$ for $\sigma > \frac{1}{1-\alpha}$.

On the hand side, if the elasticity of substitution is small, the capital reallocation effect can not offset the weaker credit easing effect so that $g_b > g_{1b}$ for $\sigma < \frac{1}{1-\alpha}$. In the borderline case with $\sigma = \frac{1}{1-\alpha}$, the positive capital reallocation effect fully offsets the weaker credit easing effect so that $g_b = g_{1b}$.

Now consider the case in which bubbles appear only in sector 2. In this case, the credit easing effect is weaker than that in the case where bubbles appear in both sectors. In addition, capital is reallocated toward the less productive sector 2. Hence the capital reallocation effect is negative. The overall effects make $g_{2b} < g_b$.

Compared to the bubbleless equilibrium, bubbles in sector 2 have a positive credit easing effect and a negative capital reallocation effect. When the elasticity of substitution between the two types of capital goods is large enough ($\sigma > \frac{1}{1-\alpha}$), the negative capital reallocation effect dominates the positive credit easing effect so that $g_{2b} < g^*$. On the other hand, when $\sigma < \frac{1}{1-\alpha}$, the negative capital reallocation effect is dominated so that $g_{2b} > g^*$. In the borderline case with $\sigma = \frac{1}{1-\alpha}$, the two effects fully offset each other so that $g_{2b} = g^*$.

Finally, we provide some numerical examples to illustrate Proposition 7. We set the parameter values as follows: $\rho = 0.01$ which implies annualized interest rate equal to 4%; $\alpha = 1/3$, which implies labor share is $2/3; \delta = 0.025$; and $A = 0.135$. These parameter values suggest that the growth rate g_0 in the model without financial frictions is equal to 4%. The additional parameters are set as $\xi = 0.2$, $\omega = 0.1$ and $\pi = 0.1$. These parameter values imply that the bubbleless growth rate $g^* = 2.6\%$ and the growth rate with symmetrical bubbles is $g_b = 3.3\%$.

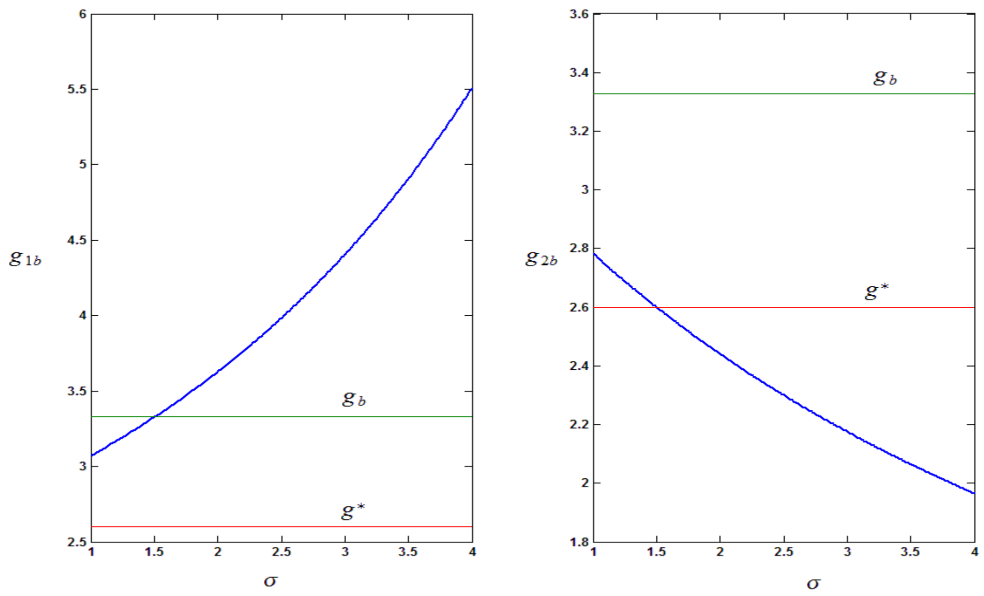

Figure 1: The effect of substitutability on growth with asymmetric bubbles

Figure 1 plots g_{1b} and g_{2b} against different values of σ. This figure reveals that g_{1b} increases with σ and g_{2b} decreases with σ. Interestingly, if σ is large enough, g_{1b} can be greater than $g_0 = 4\%$. The intuition is that even without credit constraints, capital in sector 1 is underinvested because of the externality. Bubbles in sector 1 can partially correct such a distortion by attracting more resources to that sector if the substitutability between the two types of capital is large enough.

6 Conclusion

In this paper, we provide a two-sector endogenous growth model with credit-driven stock price bubbles. These bubbles are on productive assets and occur in either one or two sectors of the economy. In addition, their occurrence is often accompanied with credit booms. Endogenous growth is driven by the positive externality effect of one type of capital goods on the productivity of workers. We show that bubbles have a credit easing effect by relaxing collateral constraints and improving investment efficiency. Sectoral bubbles also have a capital reallocation effect in that bubbles in one sector attracts more capital to be allocated to that sector. Their impact on economic growth depends on the interplay between these two effects. If the elasticity of substitution between the two types of capital goods is relatively large, then the capital reallocation effect dominates the credit easing effect. In this case, the existence of bubbles in the sector that does not generate externality will reduce long-run growth. If the elasticity is relatively small, then an opposite result holds. Bubbles may occur in the other sector that generates positive externality or in both sectors. In these cases, the existence of bubbles enhances economic growth.

In actual economies, bubbles eventually burst. Miao and Wang (2011) analyze the consequence of bubble bursting, using a one-sector model without endogenous growth. They show that the collapse of bubbles leads to a recession and moves the economy from a "good" equilibrium to a "bad" one. The present paper does not analyze this issue because this requires us to study the transitional dynamics from the equilibrium with bubbles to the equilibrium without bubbles. This analysis is technically complex and is left for a future study.[11] Nonetheless, we may provide an informal discussion here. After the collapse of bubbles, the economy will move from the balanced growth path with bubbles to the balanced growth path without bubbles characterized in Proposition 3. By Proposition 7, we can deduce that the collapse of bubbles will reduce long-run growth, except for the case in which bubbles occur in the sector without externality and in which the elasticity of substitution is large.

What are the policy implications of our model? Bubbles have a credit easing effect, which improves investment efficiency. However, sectoral bubbles also have a capital reallocation effect. In addition, the collapse of bubbles tightens credit constraints and may reduce long-run economic growth. Thus, it is important to prevent the occurrence of bubbles in the first place, rather than to prick them after their occurrence. From Proposition 3, we know that if the credit condition is sufficiently good, then bubbles cannot exist. Thus, improving credit markets is crucial to prevent the occurrence of credit-driven bubbles. In addition, as Mishkin (2008) argues, a regulatory response could be appropriate to prevent feedback loops between bubbles and the credit system.

[11] See Mulligan and Sala-i-Martin (1993) for an analysis of transitional dynamics for the Lucas (1988) model.

A Appendix: Proofs

Proof of Proposition 1: We write the Bellman equation for (8) as:

$$\begin{aligned} r_t V_i(K_{it}^j, S_{it}) &= \max_{I_t^j} \; R_{it}K_{it}^j - \pi I_{it}^j + \pi \left[V_{it}(K_{it}^j + I_{it}^j, S_{it}) - V_{it}(K_{it}^j, S_{it}) \right] \\ &\quad -\delta K_{it}^j V_{iK}(K_{it}^j, S_{it}) + V_{iS}(K_{it}^j, S_{it})\dot{S}_{it}, \end{aligned}$$

subject to (9) and (10). We use $S_{it} = (Q_{it}, B_{it})$ to denote the aggregate state vector that is independent of the firm-specific superscript j.

Substituting the conjectured form of the value function in (16) into the above Bellman equation, we obtain:

$$r_t \left(Q_{it}K_{it}^j + B_{it} \right) = \max_{I_t^j} \; R_{it}K_{it}^j - \pi I_{it}^j + \pi Q_{it} I_{it}^j - \delta K_{it}^j Q_{it} + K_{it}^j \dot{Q}_{it} + \dot{B}_{it}. \tag{65}$$

Given $Q_{it} > 1$, the investment constraint (9) and the collateral constraint (10) bind. We then obtain equation (18). As a result, the Bellman equation becomes:

$$\begin{aligned} r_t \left(Q_{it}K_{it}^j + B_{it} \right) &= R_{it}K_{it}^j + \pi \left(Q_{it} - 1 \right) \left[\left(R_{it} + \xi Q_{it} \right) K_{it}^j + B_{it} \right] \\ &\quad -\delta K_{it}^j Q_{it} + K_{it}^j \dot{Q}_{it} + \dot{B}_{it}. \end{aligned}$$

Matching coefficients on K_{it}^j and other term not involving K_{it}^j on the two sides of the equation yields equations (19) and (20) respectively. Q.E.D.

Proof of Proposition 2: It follows from Proposition 1 by integrating over $j \in [0,1]$. Q.E.D.

Proof of Proposition 3: We conjecture that (16) holds and set $B_{it} = 0$ for all t and $i = 1, 2$. Then equation (65) holds.

(i) We first suppose that the investment constraint (9) and the collateral constraint (17) do not bind. We then solve for the balanced growth rate g_0 and impose conditions on the primitives so that the supposition is verified in equilibrium. Without financial frictions, $Q_{it} = 1$ and equation (65) implies that $R_{1t} = R_{2t} = r_t + \delta$. Equating (13) with (14) yields equation (27). Substituting (27) back into (13) and (14) yields equation (26). On the balanced growth path, equation (1) becomes $r = g_0 + \rho$. It follows that

$$g_0 + \rho + \delta = r + \delta = R^* = \alpha A.$$

We then obtain equation (35). By equation (23),

$$g_0 = \frac{\dot{K}_{it}}{K_{it}} = -\delta + \pi \frac{I_{it}}{K_{it}}.$$

Solving yields:

$$\frac{I_{it}}{K_{it}} = \frac{\alpha A - \rho}{\pi}.$$

The investment constraint (9) and the collateral constraint (17) imply that

$$\frac{I_{it}}{K_{it}} \leq R^* + \xi = \alpha A + \xi.$$

For this constraint not to bind on the balanced growth path, we must have

$$\frac{\alpha A - \rho}{\pi} < \alpha A + \xi.$$

We then obtain condition (34).

(ii) Suppose condition (36) holds. Then the investment and collateral constraints bind. We only need to show that $R_{1t} = R_{2t} = R^*$ and $Q_{1t} = Q_{2t} = Q^* > 1$ along the balanced growth path. The rest of the proof follows from the analysis presented in Section 3.3. Equation (30) implies that $(\xi Q_{it} + R_{it})$ is equal to the same constant for $i = 1, 2$. Denote this constant by x. On the balanced growth path, equation (19) becomes:

$$rQ_{it} = R_{it} + x\pi \left(Q_{it} - 1\right) - \delta Q_{it}.$$

Combing with equation $\xi Q_{it} + R_{it} = x$, we can solve for Q_{it} and R_{it} :

$$Q_{it} = \frac{x\left(1-\pi\right)}{\xi + r - x\pi}, \quad R_{it} = x - \frac{\xi x\left(1-\pi\right)}{\xi + r - x\pi}.$$

Thus, Q_{it} and R_{it} are equal to some constants independent of t and i. Q.E.D.

Proof of Proposition 4: Plugging equations (40), (42), and (44) into equation (43), we can derive the growth rate g_b in (47). Substituting the expressions for Q_b, R_b, and g_b in (42), (44), and (47), respectively, into equation (45) yields:

$$\frac{B_{it}}{K_{it}} = \frac{\alpha A(1-\pi)}{\rho + \pi} - \frac{\rho}{\pi} - \xi.$$

Condition (46) ensures that $B_{it}/K_{it} > 0$.

Using equations (47) and (38), we obtain:

$$g_b - g^* = \frac{\alpha A \rho \pi \left(1-\pi\right)\left(1-\xi\right)}{\left(\rho+\pi\right)\left(\rho + \pi\xi\right)} - (1-\xi)\rho. \tag{66}$$

It follows from condition (46) that $g_b > g^*$. Q.E.D.

Proof of Proposition 5: We need to show the existence of $B_{1t}/K_{1t} > 0$ using equation (55). Comparing with equation (56), we only need to show that

$$\xi Q_1 + R_1 < \xi Q_2 + R_2 = \frac{1}{\pi}(g_{1b} + \delta) \tag{67}$$

Substituting the expressions in equations (50) and (52) for Q_1 and R_1, respectively, into the above inequality, we find that it is equivalent to (60). Q.E.D.

Proof of Proposition 6: The proof is similar to that for Proposition 5. Q.E.D.

Proof of Proposition 7: (i) Suppose $\sigma > 1/(1-\alpha) > 1$. We first show that $g_{1b} > g_b$. For the asymmetric bubbly equilibrium with $B_{1t} > 0$ and $B_{2t} = 0$, we can show that $R_1 < R_2$ as discussed in Section 5.1. It follows from equation (54) that

$$\begin{aligned} R_1 &= A\alpha\omega^{\frac{1}{\sigma}}\omega^{\alpha-1}\left[\omega^{\frac{1}{\sigma}} + \frac{1-\omega}{\omega^{\frac{\sigma-1}{\sigma}}}\left(\frac{R_1}{R_2}\right)^{\sigma-1}\right]^{\frac{\alpha\sigma-\sigma+1}{\sigma-1}} \\ &> A\alpha\omega^{\frac{1}{\sigma}}\omega^{\alpha-1}\left[\omega^{\frac{1}{\sigma}} + \frac{1-\omega}{\omega^{\frac{\sigma-1}{\sigma}}}\right]^{\frac{\alpha\sigma-\sigma+1}{\sigma-1}} = \alpha A = R_b. \end{aligned}$$

By equation (47), g_b and R_b satisfy

$$R_b = \frac{\rho(1-\xi)}{1+\rho}\frac{\rho+\pi}{\pi} + \frac{1}{\pi}\frac{\rho+\pi}{1+\rho}(\delta + g_b). \tag{68}$$

Comparing with equation (52) and using $R_1 > R_b$, we deduce that $g_{1b} > g_b$.

Next, we show that $g_{2b} < g^*$. For the asymmetric bubbly equilibrium with $B_{2t} > 0$ and $B_{1t} = 0$, we can follow a similar analysis to show that $R_2 < R_1 < \alpha A$. Using equation (61), we deduce that

$$g_{2b} < \frac{\alpha A\pi(\rho+\xi)}{\rho + \pi\xi} - \delta = g^*. \tag{69}$$

Proposition 3 shows that $g^* < g_b$. Combining the above results, we obtain that $g_{2b} < g^* < g_b < g_{1b}$.

(ii) Suppose that $0 < \sigma < 1/(1-\alpha)$. For the asymmetric bubbly equilibrium with $B_{1t} > 0$ and $B_{2t} = 0$, we know that $R_1 < R_2$. It follows from equation (54) that

$$\begin{aligned} R_1 &= A\alpha\omega^{\frac{1}{\sigma}+\alpha-1}\left[\omega^{\frac{1}{\sigma}} + \frac{1-\omega}{\omega^{\frac{\sigma-1}{\sigma}}}\left(\frac{R_1}{R_2}\right)^{\sigma-1}\right]^{\frac{\alpha\sigma-\sigma+1}{\sigma-1}} \\ &< A\alpha\omega^{\frac{1}{\sigma}+\alpha-1}\left[\omega^{\frac{1}{\sigma}} + \frac{1-\omega}{\omega^{\frac{\sigma-1}{\sigma}}}\right]^{\frac{\alpha\sigma-\sigma+1}{\sigma-1}} = \alpha A = R_b. \end{aligned}$$

Following a similar argument in the analysis in case (i), we deduce that $g_{1b} < g_b$.

We next show that $g^* < g_{1b}$. For the asymmetric bubbly equilibrium with $B_{1t} > 0$ and $B_{2t} = 0$, we plug equation (53) into equation (14) to derive

$$\begin{aligned} R_2 &= A\alpha(1-\omega)^{\frac{1}{\sigma}}\omega^{\alpha-1}K_{1t}^{1-\alpha}\left[\omega^{\frac{1}{\sigma}}K_{1t}^{\frac{\sigma-1}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}}K_{2t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\alpha\sigma-\sigma+1}{\sigma-1}} K_{2t}^{-\frac{1}{\sigma}} \quad (70) \\ &= A\alpha(1-\omega)^{\frac{1}{\sigma}}\omega^{\alpha-1}\left[\frac{\omega}{1-\omega}\left(\frac{R_2}{R_1}\right)^{\sigma}\right]^{1-\alpha}\left[\omega\left(\frac{R_2}{R_1}\right)^{\sigma-1}(1-\omega)^{\frac{1-\sigma}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}}\right]^{\frac{\alpha\sigma-\sigma+1}{\sigma-1}} \end{aligned}$$

It follows from $R_2 > R_1$ that

$$\begin{aligned} R_2 &> A\alpha(1-\omega)^{\frac{1}{\sigma}}\omega^{\alpha-1}\left[\frac{\omega}{1-\omega}\right]^{1-\alpha}\left[\omega(1-\omega)^{\frac{1-\sigma}{\sigma}} + (1-\omega)^{\frac{1}{\sigma}}\right]^{\frac{\alpha\sigma-\sigma+1}{\sigma-1}} \\ &= \alpha A \end{aligned}$$

Using equation (59), we can show that the growth rate g_{1b} and R_2 satisfy

$$g_{1b} = -\delta + \frac{\pi(\rho+\xi)}{\pi\xi+\rho}R_2 > -\delta + \frac{\pi(\rho+\xi)}{\pi\xi+\rho}\alpha A = g^*. \quad (71)$$

Thus, we obtain that $g^* < g_{1b} < g_b$.

Now, we consider the asymmetric bubbly equilibrium with $B_{2t} > 0$ and $B_{1t} = 0$. In this case, $R_1 > R_2$. As before, we can show that $R_1 > \alpha A$. By equation (70), $R_2 < \alpha A$. Using equation (62), we can show that

$$g_{2b} < \frac{\alpha A(1+\rho)\pi}{\rho+\pi} + \rho\xi - \rho - \delta = g_b.$$

Next, we show that $g_{2b} > g^*$. By equation (61), we deduce that

$$g_{2b} = -\delta + \frac{\pi\left(\rho+\xi\right)}{\pi\xi+\rho}R_1 > -\delta + \frac{\pi\left(\rho+\xi\right)}{\pi\xi+\rho}\alpha A = g^*.$$

Thus, $g^* < g_{2b} < g_b$.

(iii) From the above analysis, we can easily deduce the result when $\sigma = 1/\left(1-\alpha\right)$ in the proposition. Q.E.D.

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