Speculative Bubbles and Financial Crisis*

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Abstract

Asset prices are widely believed to be much more volatile and often detached from their fundamentals. It is also widely believed that the bursting of financial bubbles can depress the real economy. This paper addresses these issues by constructing an infinite-horizon incomplete-market general-equilibrium model with speculative bubbles. We characterize conditions under which storable goods, regardless of their intrinsic values, can carry bubbles and agents are willing to invest in such bubbles despite their positive probability of bursting. We show that systemic risk—perceived changes in the bubbles' probability to burst—can generate boom-bust cycles with hump-shaped output dynamics, and produce asset price movements that are many times more volatile than the economy's fundamentals, as in the data.

Keywords: Asset Price Volatility, Financial Crisis, Speculative Bubbles, Sunspots, Tulip Mania.

JEL Codes: E21, E22, E32, E44, E63.

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1 Introduction

It is widely believed that the current world-wide financial crisis was caused by the burst of a housing-market bubble in the United States. However, a situation like this is not new. History has too often witnessed the rise and collapse of similar asset "bubbles". The first recorded such bubble is the "Tulip mania"—a period in Dutch history during which contract prices for tulip bulbs reached extraordinarily high levels and then suddenly collapsed. At the peak of the tulip mania in February 1637, tulip contracts sold for more than 10 times the annual income of a skilled craftsman, which is above the value of a furnished luxury house in seventeenth-century Amsterdam.[1] Figure 1 shows the tulip price index during the 1636-37 period.[2]

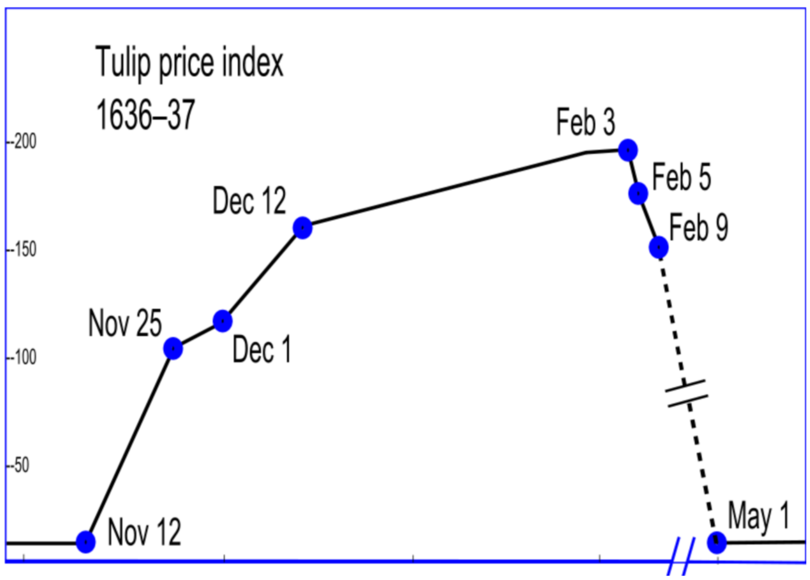

Figure 1. The Tulip mania bubble.

According to Mackay (1841, p. 107), during the tulip mania, people sold their other possessions to speculate in the tulip market:

> ... [T]he population, even to its lowest dregs, embarked in the tulip trade.... Many individuals grew suddenly rich. A golden bait hung temptingly out before the people, and, one after the other, they rushed to the tulip marts, like flies around a honey-pot. Every one imagined that the passion for tulips would last for ever, and that the wealthy from every part of the world would send to Holland, and pay whatever prices were asked for them. The riches of Europe would be concentrated on the shores of the Zuyder Zee, and poverty banished from the favoured clime of Holland. Nobles, citizens, farmers, mechanics, seamen, footmen, maidservants, even chimney-sweeps and old clotheswomen, dabbled in tulips.

People were purchasing tulips at higher and higher prices, intending to resell them for a profit. However, such a scheme could not last because tulip prices were growing faster than income. Sooner or later traders

[1] See, for example, Mackay (1841).
[2] Source: Wikipedia (http://en.wikipedia.org/wiki/Tulip_mania) and Thompson (2007).

would no longer be able to find new buyers willing to pay increasingly inflated prices. As this realization set in, the demand for tulips collapsed and the bubble burst. The Dutch economy went into a deep recession in 1637.

Although historians and economists continue to debate whether the tulip mania was indeed a bubble caused by what Mackay termed "Extraordinary Popular Delusions and the Madness of Crowds" (see, e.g., Dash, 1999; Garber, 1989, 1990; and Thompson, 2007), many observers believe that bubbles are important elements of real-world asset markets. Yet, despite the widespread belief in the existence of bubbles in the real world, it has been known, at least since the work of Scheinkman (1977, 1988), that it is difficult to construct infinite-horizon model economies in which asset price bubbles as an intertemporal equilibrium exist. Santos and Woodford (1997) characterize conditions for the existence of asset price bubbles in a general class of intertemporal equilibrium models and they show that the conditions under which bubbles are possible are quite fragile.[3]

Despite the fragile conditions acknowledged by the existing literature, it is nonetheless worthwhile to investigate if rational bubbles can help explain the business cycle. This paper constructs asset price bubbles in an infinite-horizon model with incomplete financial markets and short-sales constraints. It is shown that genuine asset bubbles with prices far exceeding the assets' fundamental values and with movements similar to Figure 1 can be constructed. In the model, infinitely lived agents are willing to invest in bubbles even though they may burst at any moment. The reason is that with incomplete financial markets and borrowing constraints, bubbles provide liquidity by serving as liquid assets. We show that the burst of such bubbles can generate recessions, and the perceived changes in the probability of the bubbles' burst can cause asset price movements many times more volatile than aggregate output.

People invest in bubbles for many reasons. The idea that infinitely lived rational agents are willing to hold bubbles with no intrinsic values to self-insure against idiosyncratic risk can be traced back at least to Bewley (1980) and Lucas (1980).[4] This idea is applied more recently in general equilibrium models by Kiyotaki and Moore (2008) and Kocherlakota (2009) to study economic fluctuations, where heterogeneous firms use intrinsically worthless assets to improve resource allocation and investment efficiency when financial markets are incomplete.[5] This paper builds on this literature to study asset price volatility and rational bubbles that may grow on storable goods with intrinsic values. This extension is not trivial because sunspot equilibrium may disappear in the Kiyotaki-Moore-Kocherlakota model once the object supporting the bubble (e.g., land) is allowed to have positive fundamental values (e.g., utilities). Casual observation suggests that more often bubbles are believed to exist in goods with positive fundamental values, such as antiques, bottles of wines, paintings, flower bulbs, rare stamps, houses, land, and so on. More importantly, we apply our model to quantitatively explain business-cycle fluctuations in aggregate output and asset prices, which is the main contribution of this paper.

[3] For more recent discussions on this literature, see Kocherlakota (2008).

[4] It can be traced further back to Samuelson's (1958) overlapping generations model of money. Bewley (1980) and Lucas (180) show in infinite-horizon economies that people are willing to hold money as a store of value to self-insure against idiosyncratic shocks despite money has no intrinsic values. The major difference between this literature (Bewley, 1980; Lucas, 1980) and our paper is that (i) we assume a different source of idiosyncratic uncertainty and (ii) we use a production economy instead of an endowment economy.

[5] The related literature also includes Angeletos (2007), Araujo, Pascoa, and Torres-Martinez (2005), Caballero and Krishnamurthy (2006), Farhi and Tirole (2008), Hellwig and Lorenzoni (2009), Kocherlakota (1992, 2008), Scheinkman and Weiss (1986), Tirole (1985), and Woodford (1986), among others. This literature focuses on asset price bubbles and financial market frictions and differ from the indeterminacy literature of Benhabib and Farmer (1994) and Wang and Wen (2007, 2008). For the earlier literature on sunspots, see Cass and Shell (1983) and Azariadis (1981).

We use a DSGE model to characterize conditions for the existence of rational bubbles that grow on goods with fundamental values. We show that any inelastically supplied storable goods,[6] regardless of their intrinsic values, can support bubbles with the following features: (i) the market price of the goods exceeds their fundamental values and (ii) the market values can collapse to fundamental values with positive probability (namely, the fundamental value is itself a possible equilibrium).[7]

The basic structure of our model closely resembles that of Kiyotaki and Moore (2008) and Kocherlakota (2009) wherein firms, instead of households, invest in bubbles. The main differences between our model and the literature include the following:

1. In addition to characterizing general equilibrium conditions for bubbles to develop on objects with positive fundamental values, in our model the probability of capital investment is endogenously determined by firms rather than exogenously fixed. That is, the portion of firms optimally choosing to invest in fixed capital each period is endogenous in our model. Hence, in equilibrium the number of firms that are investing can respond to aggregate shocks and monetary policy. This extensive margin is missing from the literature.

2. We introduce multiple assets in the model. Our multiple asset approach allows us to construct stochastic sunspot equilibrium featuring systemic risk (i.e., the probability of the bursting of bubbles perceived by the public) and conduct impulse response analyses and time-series simulations.

3. We focus on asset price volatility and show that our model can generate hump-shaped output dynamics and match the asset-price movements of the U.S. economy.

4. We provide an analytically tractable method to solve the general-equilibrium paths of our model (without resorting to numerical computational techniques as in Krusell and Smith, 1998) despite a continuum of heterogeneous agents with irreversible investment and borrowing constraints.[8]

The rest of the paper is organized as follows. Section 2 presents a basic model and characterizes conditions under which bubbles can grow on goods with intrinsic values. Section 3 introduces sunspot shocks to a version of the basic model (by allowing the perceived probability of bubbles to burst to be stochastic) and calibrates the model to match the U.S. business cycles and asset price volatility. Section 4 concludes the paper.

2 The Benchmark Model

We consider an infinite horizon economy. Time is discrete and indexed by $t = 0, 1, 2, ...$ There is a unit mass of continuum of heterogeneous firms indexed by $i \in [0, 1]$. Each firm is subject to an aggregate labor-augmenting productivity shock and an idiosyncratic investment-specific productivity shock. Households are identical and they trade firms' shares.

Assume that in the beginning of time ($t = 0$) there exists one unit of divisible good endowed from nature and equally distributed among the firms. We call the good "tulips" throughout the paper. Each unit of

[6] Goods can be producible yet at the same time inelastically supplied. For example, antiques and bottles of wine are produced goods, but their dates of production make them unique and nonsubstitutable by newly produced ones.

[7] If the fundamental value is not an equilibrium, then bubbles will never burst and thus it may be argued that bubbles do not exist (because it is difficult to know empirically whether there is a bubble if it never bursts). Also, the existence of multiple fundamental equilibria does not imply bubbles because the asset values never exceed fundamentals in a fundamental equilibrium.

[8] Our method follows that of Wang and Wen (2009). As far as we know, the existing literature—except Wang and Wen (2009)—has not shown how to solve discrete-time models with irreversible investment and borrowing constraints analytically.

tulip can be converted into f units of consumption goods and paid to households (firm owners) as dividends. Assume that households do not have the technology to store tulips but firms do, and there exists a fixed storage cost, $\zeta \geq 0$, per unit per period for firms. Hence, f is the fundamental value of tulips.[9]

Denote the market price of tulips by q_t. Obviously, if $q_t < f$, then the demand for tulips is infinity and no firm will ever want to sell tulips (i.e., the aggregate market supply is zero). Hence, if there exists an equilibrium price for tulips, it must satisfy $q_t \geq f$. One question we are interested in answering in this paper is: Do firms have incentives to hold and invest in tulips when $q_t > f$? In other words, can $q_t > f$ be supported as a competitive (bubble) equilibrium in the economy other than the fundamental equilibrium, $q_t^* = f$?

In what follows, we characterize the conditions under which a bubble equilibrium with the following features exists: (i) the market price of tulips exceeds f, and (ii) the market values of tulips can collapse to the fundamental value f with positive probability (namely, the fundamental value $q_t^* = f$ for all t is itself a possible equilibrium). These two features define a bubble equilibrium in this paper.[10]

2.1 Firms

Each firm i maximizes discounted dividends, $E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} d_t(i)$, where d_t denotes dividend, Λ_t the representative household's marginal utility that firms take as given (which may be stochastic), and $\beta \in (0,1)$ the time-discounting factor. The production technology of firm i is denoted by

$$y_t(i) = A_t k_t(i)^{\alpha} n_t(i)^{1-\alpha}, \qquad \alpha \in (0,1), \tag{1}$$

where A_t is an index of aggregate total factor productivity (TFP), $k_t(i)$ capital stock, and $n_t(i)$ employment. The capital stock is accumulated according to the law of motion, $k_{t+1}(i) = (1-\delta)k_t(i) + \frac{i_t(i)}{\varepsilon_t(i)}$, where investment is irreversible, $i_t(i) \geq 0$, and is subject to an idiosyncratic rate of return (cost) shock, $\varepsilon_t(i)$, with support, $[\underline{\varepsilon}, \bar{\varepsilon}] \in R^+ \cup \{+\infty\}$, and the cumulative distribution function, $F(\varepsilon)$. Firms make investment decisions in each period after observing $\varepsilon_t(i)$.

Firms pay dividends to the owners. A firms' dividend is defined as

$$d_t(i) = A_t k_t(i)^{\alpha} n_t(i)^{1-\alpha} + q_t h_t(i) - i_t(i) - (q_t + \zeta)\, h_{t+1}(i) - w_t n_t(i), \tag{2}$$

where w_t is the real wage, h_{t+1} the quantity (or shares) of tulips purchased in the beginning of period t as a store of value, and ζh_{t+1} the total fixed storage costs paid for storing tulips within period t. In addition, we impose the following constraints: $d_t(i) \geq 0$ and $h_{t+1}(i) \geq 0$. That is, firms can neither pay negative dividends nor hold negative amounts of tulips. These assumptions imply that firms are financially constrained and the asset markets are incomplete. Such constraints plus investment irreversibility may give rise to speculative (precautionary) motives for investing in tulip bubbles.

To characterize firms' optimization program, the following steps simplify our analysis. Using the firm's optimal labor demand schedule,

$$(1-\alpha) A_t k_t(i)^{\alpha} n_t(i)^{-\alpha} = w_t, \tag{3}$$

[9] For simplicity, assume that tulips cannot be used as a factor of production.

[10] That is, our definition of a bubble equilibrium requires that the fundamental value of an asset itself be an equilibrium so that the burst of bubbles is possible.

we can express labor demand as a linear function of the capital stock, $k_t(i)$,

$$n_t(i) = \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1}{\alpha}} k_t(i). \tag{4}$$

Accordingly, output $y_t(i)$ is also a linear function of $k_t(i)$,

$$y_t(i) = A_t \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} k_t(i) \tag{5}$$

These linear relations imply that aggregate output and employment may depend only on the aggregate capital stock. Thus, we do not need to track the distribution of $k_t(i)$ to study aggregate dynamics. Defining $R_t \equiv \alpha A_t \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}}$, the firm's net revenue is given by

$$y_t(i) - w_t n_t(i) = R_t k_t(i), \tag{6}$$

which is also linear in the capital stock.

Using the definition of R_t, the firm's problem is to solve

$$V_0(i) = \max E_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} d_t(i)\right], \tag{7}$$

where $d_t(i) = [R_t k_t(i) - i_t(i) + q_t h_t(i) - (q_t + \zeta) h_{t+1}(i)]$, subject to the following constraints:

$$d_t(i) \geq 0 \tag{8}$$

$$h_{t+1}(i) \geq 0 \tag{9}$$

$$i_t(i) \geq 0 \tag{10}$$

$$k_{t+1}(i) = (1-\delta) k_t(i) + \frac{i_t(i)}{\varepsilon_t(i)}, \tag{11}$$

given $k_0(i) > 0$ and $h_0(i) = 1$. Here, $\beta^t \Lambda_t / \Lambda_0$ is the stochastic discount factor between period 0 and t. We will show later that Λ_t is the representative household's marginal utility in period t.

2.2 Household

All household are identical. A representative household chooses consumption C_t, labor supply N_t, and share holdings of different firms $s_{t+1}(i)$ to maximize life-time utility,

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - A_n \frac{N_t^{1+\gamma_n}}{1+\gamma_n}\right), \quad \beta \in (0,1); \tag{12}$$

subject to the budget constraint

$$C_t + \int_0^1 s_{t+1}(i) \left[V_t(i) - d_t(i)\right] di = \int_0^1 s_t(i) V_t(i) di + w_t N_t. \tag{13}$$

In addition, households cannot short tulips. Defining Λ_t as the Lagrangian multiplier of equation (13), the first-order conditions for $\{s_{t+1}(i), C_t, N_t\}$ are given, respectively, by

$$V_t(i) = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}(i) \right\} + d_t(i), \tag{14}$$

$\frac{1}{C_t} = \Lambda_t$, and $A_n N_t^{\gamma_n} = \Lambda_t w_t$. Equation (14) implies that the stock price $V_t(i)$ is given by the discounted present value of dividends as in equation (7). Also, Λ_t is the marginal utility of consumption.

2.3 Competitive Equilibrium

A competitive equilibrium is defined as the sequences of quantities, $\{i_t(i), n_t(i), k_{t+1}(i), y_t(i), h_t(i), s_{t+1}(i)\}_{t\geq 0}$ and $\{C_t, N_t\}_{t\geq 0}$, and prices $\{w_t, q_t, V_t(i)\}_{t\geq 0}$ for $i \in [0,1]$, such that:

(i) Given prices $\{w_t, q_t\}_{t\geq 0}$, the sequence of quantities $\{i_t(i), n_t(i), k_{t+1}(i), h_{t+1}(i), x_t(i)\}_{t\geq 0}$ solves each firm i's problem (7) subject to the constraints (8) through (11).

(ii) Given prices $\{w_t, V_t(i)\}_{t\geq 0}$, the sequence $\{C_t, N_t, s_t(i)\}_{t\geq 0}$ maximizes household utility (12) subject to the budget constraint (13).

(iii) All markets clear:

$$s_t(i) = 1 \quad \text{for all } i \in [0,1] \tag{15}$$

$$N_t = \int_0^1 n_t(i)di \tag{16}$$

$$C_t + \int_0^1 \frac{i_t(i)}{\varepsilon_t(i)}di + \zeta \int_0^1 h_{t+1}(i)di = \int_0^1 y_t(i)di + f x_t, \tag{17}$$

where the first equation pertains to the shares market, the second to the labor market, and the third to the aggregate goods market. The third term on the LHS of equation (17) is the total storage costs and the second term on the RHS of equation (17), $x_t \equiv \left[\int_0^1 h_t(i)di - \int_0^1 h_{t+1}(i)di\right]$, is the total number of tulips taken out of circulation from the tulip market and converted to consumption goods in period t. As will be shown below, if $q_t > f$, then $x_t = 0$.

2.4 Decision Rules

This subsection shows how to derive firms' optimal decision rules in closed forms. It should be noted that the main simplification comes from the linearality of firms' objective functions and constraints and the fact that firms can take the household's marginal utility and other market prices as given.

Proposition 1 *A firm's optimal decision rule for fixed investment is given by*

$$i_t(i) = \begin{cases} R_t k_t(i) + q_t h_t(i) & \varepsilon_t(i) \leq \varepsilon_t^* \\ & \\ 0 & \varepsilon_t(i) > \varepsilon_t^* \end{cases}, \tag{18}$$

where ε_t^* is a time-varying cutoff that is independent of i and is determined by the following Euler equation:

$$\varepsilon_t^* = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[R_{t+1} Q(\varepsilon_{t+1}^*) + (1-\delta)\, \varepsilon_{t+1}^* \right], \tag{19}$$

where $Q(\cdot) > 1$ captures the option value (liquidity premium) of one unit of cash flow and this option value is determined by

$$Q(\varepsilon_t^*) = \int \max\left\{1, \frac{\varepsilon_t^*}{\varepsilon_t(i)}\right\} dF(\varepsilon). \tag{20}$$

When the aggregate demand for tulips $\Omega_{t+1} \equiv \int_0^1 h_{t+1}(i)di > 0$, the equilibrium tulip price is determined by the following asset pricing equation:

$$q_t + \zeta = \beta E_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} q_{t+1} Q(\varepsilon_{t+1}^*)\right]. \tag{21}$$

Proof. See Appendix I. ■

The intuition behind proposition 1 is as follows. First, the marginal cost of investment is 1 and the marginal gain of investment is $\frac{1}{\varepsilon_t(i)} \times \lambda_t$, where λ_t is the market value of one unit of newly installed capital. Thus, the firm-specific Tobin's q equals $\frac{\lambda_t}{\varepsilon_t(i)}$ and the firm will invest if and only if $\frac{\lambda_t}{\varepsilon_t(i)} \geq 1$ or $\varepsilon_t(i) \leq \lambda_t$. Therefore, the cutoff is given by $\varepsilon_t^* \equiv \lambda_t$. Since the market value of newly installed capital is determined by expected future marginal products of capital, the cutoff is independent of i because $\varepsilon_t(i)$ is i.i.d.

Second, the value of one unit of cash flow (Q_t) is greater than 1 because of the option of waiting. When the cost of capital investment is low in period t ($\varepsilon_t(i) \leq \varepsilon_t^*$), one dollar of cash flow yields $\frac{\varepsilon^*}{\varepsilon(i)} > 1$ dollars of new capital through investment. When the cost is high in period t ($\varepsilon_t(i) > \varepsilon_t^*$), firms can hold on to the cash as inventories and the rate of return is simply 1. This explains why the option value of cash (Q_{t+1}) enters the right-hand side of equations (19) and (21) as part of future returns to reflect a liquidity premium of cash.

Third, equation (19) is the Euler equation for capital investment. The left-hand side (LHS) is the market price of one unit of newly installed capital. The right-hand side (RHS) is the expected marginal gains of having one unit of newly installed capital, which includes two terms: (i) one unit of new capital can generate R units of outputs the next period with option value RQ; and (ii) it has a residual market value $(1-\delta)\,\varepsilon_{t+1}^*$ next period after depreciation. A firm will invest up to the point where the LHS equals the RHS. This equation also determines the optimal cutoff (ε_t^*) in the model.

Forth, equation (21) is the Euler equation for tulip investment. The LHS is the marginal cost of holding one unit of tulips, which includes the market price q and storage cost ζ. The RHS is the expected next-period gain by having one unit of tulip in hand. Because the liquidity premium of cash flow $Q > 1$, having tulips in hand can improve firms' cash positions through liquidation in the case of cash shortage. Thus, the effective rate of return to tulip assets the next period is $q_{t+1}Q_{t+1}$. This equation is thus the asset pricing equation for tulips.

Intuitively, because tulips are storable for firms, they thus allow a firm to self-insure against idiosyncratic shocks by serving as a store of value (i.e., liquidity). For example, if the cost shock $\varepsilon_t(i)$ is large (or the rate of return to capital investment is low) in period t, firms may opt to invest in tulips so as to have liquidity available in the future when the next-period cost of capital investment may be low. On the other hand, if the rate of return to capital investment is high ($\varepsilon_t(i)$ is small), firms may opt to liquidate (sell) tulips in hand and make more liquidity available now to purchase fixed capital and expand production capacity. Such behavior may be rational despite the fact that the tulip price exceeds its fundamental value and tulip bubbles have a positive probability to burst.

2.5 General Equilibrium

The aggregate variables are defined as $N_t = \int_0^1 n_t(i)di$, $I_t = \int_0^1 i_t(i)di$, $K_t = \int_0^1 k_t(i)di$, and $Y_t = \int_0^1 y_t(i)di$. Given that $k_t(i)$ is a state variable, by the factor demand functions of firms we have $N_t = \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1}{\alpha}} K_t$ and $Y_t = A_t \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_t$. These two equations imply that aggregate output can be written as a simple function of aggregate labor and capital, $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$. Hence, the real wage is $w_t = (1-\alpha)\frac{Y_t}{N_t}$ and we have $R_t = \alpha \frac{Y_t}{K_t}$, which turns out to be the aggregate marginal product of capital.

Proposition 2 *The general equilibrium paths of the model are characterized by nine aggregate variables, $\{C_t, I_t, N_t, Y_t, K_{t+1}, q_t, \varepsilon_t^*, \Omega_{t+1}, \phi_t^*\}$, which are fully determined by the following system of nine nonlinear dynamic equations:*

$$Y_t = A_t {K_t}^{\alpha} N_t^{1-\alpha} \tag{22}$$

$$C_t + I_t = Y_t + f(\Omega_t - \Omega_{t+1}) - \zeta \Omega_{t+1} \tag{23}$$

$$(1-\alpha)\frac{Y_t}{C_t} = A_n N_t^{1+\gamma_n} \tag{24}$$

$$\frac{q_t + \zeta}{C_t} = \beta E_t \left\{ \frac{q_{t+1}}{C_{t+1}} Q_{t+1} \right\} + \frac{\phi_t^*}{C_t} \tag{25}$$

$$I_t = \left[\alpha Y_t + q_t \Omega_t\right] F(\varepsilon_t^*) \tag{26}$$

$$\frac{\varepsilon_t^*}{C_t} = \beta E_t \left\{ \frac{\varepsilon_{t+1}^*}{C_{t+1}} \left[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{Q_{t+1}}{\varepsilon_{t+1}^*} + 1 - \delta \right] \right\} \tag{27}$$

$$K_{t+1} = (1-\delta) K_t + \omega(\varepsilon_t^*) I_t \tag{28}$$

$$\Omega_{t+1} \phi_t^* = 0. \tag{29}$$

Proof. See Appendix II. ■

Equation (22) is the aggregate production function, equation (23) is the aggregate resource constraint derived from equation (17), equation (24) pertains to the optimal labor supply decision of the household, equation (25) is the Euler equation for optimal tulip investment based on equation (21), equation (26) characterizes the level of aggregate investment, equation (27) is the Euler equation for optimal capital investment based on equation (19), equation (28) expresses the law of motion of aggregate capital accumulation, and equation (29) is a complementarity condition that determines whether tulips are traded or not in general equilibrium—that is, $\Omega_{t+1} > 0$ only if $\phi_t^* = 0$ and $\phi_t^* > 0$ only if $\Omega_{t+1} = 0$.

Equation (26) shows that tulip assets affect aggregate investment through two channels. First, they directly increase all firms' cash flows through the liquidation value, $q_t \Omega_t$. Second, they influence the cutoff value, ε^*, thus affecting the number of active firms (that make fixed investments) along the extensive margin and, consequently, the marginal efficiency of aggregate investment. The last channel plays a critical role in our model's dynamics but is absent in the models of Kiyotaki and Moore (2008) and Kocherlakota (2009).

There are two possible steady states in the model. In one steady state, tulips are never consumed and their market price is greater than or equal to their fundamental value—namely, $q \geq f$, $\Omega = 1$, and $\phi^* = 0$. In the other steady state, the market price equals the fundamental value and tulips are not circulated among firms—namely, $q = f$, $\Omega = 0$, and $\phi^* \geq 0$. We are now ready to characterize the nature of these steady states.

Steady State A: $q \geq f$, $\Omega = 1$, and $\phi^* = 0$. In this steady state, equation (23) and equations (25) through (28) become

$$C + I = Y - \zeta \tag{30}$$

$$q + \zeta = \beta q Q(\varepsilon^*) \tag{31}$$

$$I = (\alpha Y + q)\, F(\varepsilon^*) \tag{32}$$

$$1 - \beta\,(1-\delta) = \beta\alpha \frac{Y}{K} \frac{Q(\varepsilon^*)}{\varepsilon^*} \tag{33}$$

$$\delta K = \omega(\varepsilon^*) I. \tag{34}$$

Equations (33) and (34) solve for the capital-to-output ratio ($\frac{K}{Y}$) and the saving rate ($\frac{I}{Y}$) given the cutoff ε^*. Equation (30) then determines the consumption-to-output ratio. Equation (24) and the production function then determine the levels of aggregate output and employment and hence the levels of consumption and investment. Equations (31) and (32) then jointly determine the cutoff (ε^*) and the asset price (q). Notice that equation (31) suggests $Q(\varepsilon^*) > 1$; hence, an interior solution for the cutoff $\varepsilon^* \in [\underline{\varepsilon}, \bar{\varepsilon}]$ exists provided that the storage cost ζ is not too high.[11]

Steady State B: $q = f$, $\Omega = 0$, and $\phi^* \geq 0$. In this steady state, no firm will invest in tulips. Denoting a variable x with subscript b (i.e., x_b) as the value of this variable in steady state B, the first-order conditions (23) and (25) through (28) become

$$C_b + I_b = Y_b \tag{35}$$

$$f + \zeta = \beta f Q(\varepsilon_b^*) + \phi_b^* \tag{36}$$

$$I_b = \alpha Y_b F(\varepsilon_b^*) \tag{37}$$

$$1 - \beta\,(1-\delta) = \beta\alpha \frac{Y_b}{K_b} \frac{Q(\varepsilon_b^*)}{\varepsilon_b^*} \tag{38}$$

$$\delta K_b = \omega(\varepsilon_b^*) I_b. \tag{39}$$

Equations (37) through (39) imply

$$1 - \beta + \beta\delta = \beta\delta \frac{Q(\varepsilon_b^*)}{\varepsilon_b^* \int_{\varepsilon \leq \varepsilon_b^*} \varepsilon^{-1} dF}, \tag{40}$$

which uniquely solves for the cutoff ε_b^*. To see that we have an interior solution for the cutoff, $\underline{\varepsilon} < \varepsilon_b^* < \bar{\varepsilon}$, notice that the RHS of equation (40) equals infinity at $\varepsilon_b^* = \underline{\varepsilon}$ and equals $\beta\delta$ at $\varepsilon_b^* = \bar{\varepsilon}$. Hence, a unique interior solution exists as long as $\beta < 1$. Given ε_b^*, we can then solve for $\{Y_b, C_b, K_b, N_b, \phi_b^*\}$. To ensure that the condition $\phi_b^* \geq 0$ holds, equation (36) implies $\zeta \geq f\,[\beta Q(\varepsilon_b^*) - 1]$; that is, the storage cost must be large enough for steady state B to be an equilibrium. This suggests that when $\zeta = 0$ and $f > 0$, steady state B may not be possible (see Proposition 4 below).

We call steady state A a bubble steady state and steady state B a fundamental steady state. These equilibria may or may not exist, depending on parameter values. The following propositions characterize the properties and conditions for these steady states to exist.

Proposition 3 *The cutoff value (ε^*) is lower in steady state A than in steady state B. Consequently, there*

[11] Since $Q(\varepsilon^*)$ is bounded above by $Q(\bar{\varepsilon})$, if ζ is too high, then equation (31) becomes an inequality, $q + \zeta > \beta q Q(\bar{\varepsilon})$; hence, no firm will hold tulips.

are fewer firms investing in fixed capital in steady state A than in steady state B, and the marginal efficiency of aggregate investment is higher in a bubble equilibrium (steady state A) than in a no-bubble equilibrium. As a result, the aggregate capital stock-to-output ratio is higher in the bubble equilibrium than in the no-bubble equilibrium. However, the aggregate investment to output ratio is not necessarily higher in the bubble equilibrium.

Proof. See Appendix III. ■

The intuition behind proposition 3 is as follows. The first best allocation in our model is to have only the most productive firm (i.e., the firm with $\varepsilon_t(i) = \underline{\varepsilon}$) to invest in fixed capital and all other firms to lend to the most efficient firm. However, due to market incompleteness and the inability to share risk across firms, less efficient firms also opt to invest in fixed capital despite high costs (i.e., high realizations of $\varepsilon_t(i)$). Because tulips provide a self-insurance device that enables firms to intertemporally transfer funds and thus reduce borrowing constraints, a bubble equilibrium reduces the need for less efficient firms to invest in fixed capital (because the alternative of investing in tulips may yield higher expected returns when $\varepsilon_t(i)$ is sufficiently high). Therefore, the fraction of firms investing in fixed capital is lower in a bubble steady state than in a no-bubble steady state. Since the number of firms investing is determined by the cutoff (i.e., a firm will invest if and only if $\varepsilon_t(i) \leq \varepsilon^*$), the existence of tulip bubbles also lowers the cutoff. However, although the number of capital-investing firms is smaller in a bubble equilibrium, the average efficiency of investment is improved (because the low efficiency firms drop out). This improvement in aggregate investment efficiency enables the economy to have both higher consumption (C) and capital stock (K) with the same amount of total investment expenditure (I). Nonetheless, total investment expenditure to output ratio could be either lower or higher in a bubble equilibrium, depending on the relative strength of the income effect (due to investment efficiency) and substitution effect (since investment crowds out consumption), which in turn depends on the model's parameter values and the distribution of $\varepsilon_t(i)$. However, we can show by impulse response analysis that aggregate investment expenditure (I_t) and the investment-to-output ratio ($\frac{I_t}{Y_t}$) always increase along the transitional path towards the steady state when more bubbles are injected into the economy (see, e.g., figure 2 in the next section).

Proposition 4 *(i) If $\zeta = 0$ and $f = 0$, then a fundamental equilibrium (steady state) exists where $q = 0$, $\Omega = 0$ and $\phi^* \geq 0$. In addition, a bubble equilibrium also coexists where $q > 0$, $\Omega = 1$ and $\phi^* = 0$ provided that β is sufficiently large. (ii) If $\zeta = 0$ and $f > 0$, then only one equilibrium exists. This equilibrium is a fundamental equilibrium with $q = f$, $\Omega = 0$ and $\phi^* \geq 0$ if agents are sufficiently impatient (i.e., β small enough); this equilibrium is a bubble equilibrium with $q \geq f$, $\Omega = 1$ and $\phi^* = 0$ if β is large enough. That is, given β (and other parameters) there cannot simultaneously exist a fundamental equilibrium and a bubble equilibrium. (iii) If $\zeta > 0$ and $f \geq 0$, then a fundamental equilibrium exists where $q = f$, $\Omega = 0$ and $\phi^* \geq 0$; and a bubble equilibrium also coexists where $q \geq f$, $\Omega = 1$ and $\phi^* = 0$ if β is large enough.*

Proof. See Appendix IV. ■

Case (i) in Proposition 4 states that multiple equilibria are possible if tulip assets have no fundamental values and agents are sufficiently patient. This is a standard result in monetary theory. Case (ii) in Proposition 4 states that when storage cost $\zeta = 0$ and the fundamental value $f > 0$, if the model's structural parameters are such that $q \geq f$ and $\Omega = 1$ is a possible equilibrium, then $q = f$ and $\Omega = 0$ cannot be an equilibrium. In other words, bubbles will never burst if tulips (or money) have intrinsic values. On the other hand, if $q = f$ and $\Omega = 0$ is an equilibrium, then there cannot be a bubble equilibrium with $q \geq f$ and $\Omega = 1$. This suggests that sunspot equilibrium does not exist in the models of Kiyotaki and Moore (2008) and Kocherlakota (2009) if land has intrinsic values but with zero or small storage costs, regardless of the value of β. This result is a bit counter-intuitive, so we provide some explanations below.

When a storable good has positive fundamental value (e.g., the utility value $f > 0$), then eating it or holding it as a store of value faces a trade-off and this trade-off depends on the expected future value of the good, $\beta q_{t+1} Q_{t+1}$, where Q is the option value of cash. Since $q \geq f > 0$, this expected future value is strictly positive even if no firm expects other firms to hold the good as an asset in the next period (that is, even if the asset is only worth f in the future). Thus, if the good is an effective device for self insurance (i.e., the option value Q is large enough so that the rate of return $\beta Q \geq 1$), all firms will opt to hold the good as an asset and this is the only equilibrium because other firms' behavior no longer matters for each firm's decision making. Hence, the aggregate demand for asset can never be zero (i.e., $\Omega = 1$). On the other hand, if β is small enough, then $\beta Q < 1$ regardless of how other firms behave, so eating the asset is the only equilibrium. Therefore, multiple equilibria are not possible if $\zeta = 0$ and $f > 0$.

An alternative explanation is to note that when the storage cost $\zeta = 0$, equation (31) and equation (36) cannot be simultaneously satisfied. In a no-bubble steady state, we have $\beta Q(\varepsilon_b^*) = 1 - \frac{\phi_b^*}{f} \leq 1$ based on equation (36). On the other hand, Proposition 3 shows that the cutoff in a bubble steady state must satisfy $\varepsilon^* < \varepsilon_b^*$. This suggests that $\beta Q(\varepsilon^*) < 1$, which is inconsistent with equation (31). Hence, if $\Omega = 0$ is an equilibrium, then $\Omega = 1$ cannot be an equilibrium, and vice versa.

Therefore, the only condition under which the two steady states can coexist when $f > 0$ is for the storage cost ζ to be sufficiently large. This explains the other cases in Proposition 4. The intuition is as follows. First, if ζ is sufficiently large and $\Omega = 0$ and $q = f$, firms do not have incentives to deviate from the fundamental equilibrium by investing in tulips because the storage cost is too high to have a large enough expected asset return $E_t \beta \frac{f}{f+\zeta} Q_{t+1}$. On the other hand, if firms expect other firms to hold tulips and the market price of tulips is sufficiently high relative to the fundamental value f and the storage cost ζ, then it may also be in their own interest to hold tulips because the expected future return, $E_t \beta \frac{q}{q+\zeta} Q_{t+1}$, is large enough.

It is straightforward to confirm by the eigenvalue method that any steady state of the model is a saddle. Hence, firm-level decisions for capital and asset investment converge to time-invariant distributions in the long run without aggregate shocks. Dynamic impulse responses of the model to fundamental shocks can thus be analyzed by standard methods in the RBC literature.

3 Systemic Risk and Asset Price Volatility

This section extends the benchmark model to explain asset price volatility in the U.S. economy by allowing the possibility for bubbles to burst (as in Kocherlakota, 2009). We introduce multiple assets and stochastic sunspot shocks to affect the probability of bubbles to burst. The idea of multiple bubble assets is akin to that in Kareken and Wallace (1981) and King, Wallace, and Weber (1992). Although the steps for deriving equilibrium conditions are similar and analogous to those in the benchmark mode, we detail some of the equations for the sake of completeness and self-containedness.

The reason for introducing multiple assets is to allow the possibility of recurrent bubbles. In the benchmark model, a bubble will never come back once it bursts. If the same bubble could come back, it would be rational to hold on to it forever—so the bubble would never burst in the first place. Our strategy to introduce recurrent bubbles is to allow new bubbles to emerge in each period. These new bubbles are different from the old bubbles only by color and we assume that producing new bubbles costs no social resources, otherwise they are identical to old bubbles.

Assume there is a continuum types of "tulips" indexed by a spectrum of colors $j \in \mathbf{R}^+$. To simplify the analysis, tulips are assumed to (i) be perfectly storable with no storage costs ($\zeta = 0$), (ii) differ only in their colors (types), and (iii) have no intrinsic values ($f = 0$).[12] Thus, according to Proposition 2, each type of tulip asset can be a bubble with the following property: (i) Its equilibrium price is zero if no firms in the economy expect other firms to invest in it; and (ii) the market price is strictly positive if all agents expect others to hold it.

In each period a constant measure z of new colors of tulips is born (issued) by nature.[13] The supply of each color (variety) of tulips is normalized to 1 and each tulip type has a unique color. Also, all tulips have the same probability $\mathbf{p}_t$ to disappear (dead or destroyed) in each period regardless of color, but these events are independent of each other. In other words, different types of assets decay independently but the probability of decaying is a common shock that hits all types of assets. This assumption captures the concept of systemic risk. The newborn tulips are distributed equally to all agents (firms) as endowments, and issuing (producing) new tulips does not cost any social resources. Let q_t^j denote the price of a tulip (with color) j and $h_{t+1}^j(i)$ the quantity of the tulip j demanded by firm $i \in [0,1]$. The aggregate number (stock) of tulips evolves over time according to the law of motion:

$$\Omega_{t+1} = (1 - \mathbf{p}_t)\Omega_t + z, \tag{41}$$

where Ω is the measure of the stock of tulips in the entire economy. The market clearing condition for each tulip with color j is

$$\int_{i=0}^{1} h_{t+1}^j(i)di = 1. \tag{42}$$

As in the benchmark model, firms have the same constant returns to scale production technologies and are hit by idiosyncratic cost shocks to the marginal efficiency of investment $\varepsilon(i)$. A firm's problem is to determine a portfolio of tulips to maximize discounted future dividends. Its resource constraint is

$$d_t(i) + i_t(i) + \int_{j\in\Omega_{t+1}} q_t^j h_{t+1}^j(i)dj + w_t n_t(i) = A_t k_t(i)^\alpha n_t(i)^{1-\alpha} + \int_{j\in\Omega_t} q_t^j h_t^j(i)\mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj, \tag{43}$$

where w is the real wage, Ω is the set of available colors of tulips (this is an abuse of notation because we also denote Ω as the measure of total tulips), and the index variable $\mathbf{1}_t^j$ satisfies

$$\mathbf{1}_t^j = \begin{cases} 1 & \text{with probability} \quad 1 - \mathbf{p}_t \\ \\ 0 & \text{with probability} \quad \mathbf{p}_t \end{cases}. \tag{44}$$

Namely, each tulip bought in period $t-1$ may lose its value completely with probability $\mathbf{p}_t$ in the beginning of period t. As in the basic model, we impose the following constraints: $i_t(i) \geq 0$, $d_t(i) \geq 0$, and $h_{t+1}^j(i) \geq 0$ for all $j \in \Omega$.

Using the same definition of R as in the benchmark model, the firm's problem is to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left[R_t k_t(i) - i_t(i) + \int_{j\in\Omega_t} q_t^j h_t^j(i)\mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj - \int_{j\in\Omega_{t+1}} q_t^j h_{t+1}^j(i)dj \right], \tag{45}$$

[12] These assumptions reduce the number of parameters and simplify our calibration analysis.
[13] We can also allow z to be a stochastic endowment shock.

subject to

$$d_t(i) \geq 0 \tag{46}$$

$$h_{t+1}^j(i) \geq 0 \quad \text{for all } j \tag{47}$$

$$i_t(i) \geq 0 \tag{48}$$

$$k_{t+1}(i) = (1-\delta)\, k_t(i) + \frac{i_t(i)}{\varepsilon_t(i)}, \tag{49}$$

where the fourth term in the objective function, $\int_{j\in z} q_t^j dj$, captures tulip injection by nature in each period.

Proposition 5 *The decision rule of firm-level investment is given by*

$$i_t(i) = \begin{cases} R_t k_t(i) + \int_{j\in\Omega_t} q_t^j h_t^j(i) \mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj & \varepsilon_t(i) \leq \varepsilon_t^* \\ \\ 0 & \varepsilon_t(i) > \varepsilon_t^* \end{cases}. \tag{50}$$

The equilibrium asset price for tulips with color j is determined by

$$q_t^j = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} q_{t+1}^j \mathbf{1}_{t+1}^j Q_{t+1}. \tag{51}$$

Proof. See Appendix V. ■

The RHS of the asset-pricing equation (51) is the expected rate of return to tulip j. This equation shows that if $\mathbf{p}_t = 1$ (i.e., $\mathbf{1}_t^j = 0$ with probability 1), then tulip j's equilibrium price is given by $q_t^j = 0$ for all t because the demand for such an asset is zero when it has no market value in the next period. More importantly, even if $\mathbf{p} < 1$ (e.g., $\mathbf{p} = 0$), $q_t^j = 0$ for all t is still an equilibrium because no firms will hold tulip j if they do not expect others to hold it. In the next section, we define restrictions on the value of $\mathbf{p}$ so that $q_t^j > 0$ constitutes a bubble equilibrium.

3.1 Aggregation and General Equilibrium

As in the benchmark model, at the aggregate level we have $N_t = \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1}{\alpha}} K_t$, $Y_t = A_t \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_t$, $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$, $w_t = (1-\alpha)\frac{Y_t}{N_t}$, and $R_t = \alpha \frac{Y_t}{K_t}$. Consider a symmetric equilibrium[14] where the prices of tulips of all colors are the same:

$$q_t^j \mathbf{1}_t^j = \begin{cases} q_t & \text{with probability} \quad 1-\mathbf{p}_t \\ \\ 0 & \text{with probability} \quad \mathbf{p}_t \end{cases}, \tag{52}$$

where q_t is the market price of tulips of all colors. Define $x_t(i) \equiv \int_{j\in\Omega_t} q_t^j h_t^j(i) \mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj$ and $X_t \equiv \int_0^1 x(i) di$. By the law of large numbers, we have $X_t = (1-\mathbf{p}_t)\,\Omega_t q_t + z q_t$. Hence, aggregate capital investment is given by $I_t = (\alpha Y_t + [(1-\mathbf{p}_t)\,\Omega_t + z]\, q_t)\, F(\varepsilon_t^*)$.

The household remains the same as in the benchmark model with $f = \zeta = 0$. The first-order conditions of the household are thus the same as before. The equilibrium paths of the model can be characterized by eight

[14] A symmetric equilibrium exists because of arbitrage across tulips of different colors.

variables, $\{C, I, N, Y, K', q, \varepsilon^*, \Omega'\}$, which are solved by the following system of eight nonlinear equations:

$$Y_t = A_t K_t{}^{\alpha} N_t^{1-\alpha} \tag{53}$$

$$C_t + I_t = Y_t \tag{54}$$

$$(1-\alpha)\frac{Y_t}{C_t} = A_n N_t^{1+\gamma_n} \tag{55}$$

$$\Omega_{t+1} = (1-\mathbf{p}_t)\Omega_t + z \tag{56}$$

$$\frac{q_t}{C_t} = \beta E_t \left\{ \frac{q_{t+1}}{C_{t+1}}(1-\mathbf{p}_{t+1})Q_{t+1} \right\} \tag{57}$$

$$I_t = \left[\alpha Y_t + \left((1-\mathbf{p}_t)\,\Omega_t + z\right) q_t\right] F(\varepsilon^*) \tag{58}$$

$$\frac{\varepsilon_t^*}{C_t} = \beta E_t \left\{ \frac{\varepsilon_{t+1}^*}{C_{t+1}} \left[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{Q_{t+1}}{\varepsilon_{t+1}^*} + 1 - \delta \right] \right\} \tag{59}$$

$$K_{t+1} = (1-\delta)\,K_t + I_t \frac{\int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF}{F(\varepsilon_t^*)}. \tag{60}$$

This system of equations are saddle-path stable; namely, the number of stable eigenvalues equals the number of predetermined state variables. The model has a bubble steady state and a fundamental steady state. The equilibrium dynamics of the model are solved by log-linearizing the above system of equations around the bubble steady state.

3.2 Stationary Sunspot Equilibria

We call tulip j a bubble if $q_t^j > 0$. When the bubble bursts, we have $q_t^j = 0$. By arbitrage, after a bubble bursts, its value must remain zero permanently, otherwise people may opt to hold it indefinitely based on speculation. In each period there are fraction $\mathbf{p}_t$ of the bubbles bursting and a measure of z new bubbles being born. Changes in $\mathbf{p}_t$ are driven by sunspots (i.e., the mood of the population), which can follow any stochastic processes. In what follows, we focus on stationary sunspot equilibria with positive and bounded asset prices ($q_t > 0$ for all t).

The sunspot equilibrium condition, $q > 0$, puts some restrictions on the values of $\mathbf{p}$. Given that $q > 0$, equation (57) implies $1 = \beta(1-\mathbf{p})Q(\varepsilon^*)$ in the steady state. Equation (59) implies $Q(\varepsilon^*) = \frac{1-\beta(1-\delta)}{\beta R}\varepsilon^*$. Together we have

$$1 - \mathbf{p} = \frac{R}{1-\beta\,(1-\delta)} \frac{1}{\varepsilon^*}. \tag{61}$$

This equation determines the cutoff value $\varepsilon^*(\mathbf{p})$ as a function of $\mathbf{p}$. This equation is clearly satisfied when $\mathbf{p} = 0$. Proposition 3 shows that the cutoff in the bubble steady state is less than the cutoff in the no-bubble steady state (ε_b^*). Hence, an interior solution for the cutoff requires $\varepsilon^*(\mathbf{p}) < \varepsilon_b^*$. Thus we must have

$$1 - \mathbf{p} > \frac{R}{1-\beta\,(1-\delta)} \frac{1}{\varepsilon_b^*}. \tag{62}$$

Notice that $\frac{R}{1-\beta(1-\delta)} = \sum_{j=0}^{\infty} \beta^j (1-\delta)^j \alpha \frac{Y}{K}$ is the present value of the marginal products of capital and $\frac{1}{\varepsilon}$ is the marginal efficiency of investment. Hence, the conditions in equation (62) state that the real expected rate of return to tulips (i.e., the survival probability of a speculative bubble) must be comparable to that of capital investment (Tobin's q) to induce people to hold both capital and bubbles simultaneously.

Since $\lim_{\varepsilon^*\to\underline{\varepsilon}} \frac{R}{1-\beta(1-\delta)}\frac{1}{\varepsilon^*}=1$, the larger the spread between the lower and upper bounds of the support $[\underline{\varepsilon},\bar{\varepsilon}]$, the larger is the permissible region for the value of $\mathbf{p}$. This simply restates the finding that uninsured idiosyncratic uncertainty in the expected rate of returns to capital investment (or Tobin's q) is the fundamental reason for people to invest in bubbles. When idiosyncratic assessments of risks converge (e.g., $\underline{\varepsilon}=\bar{\varepsilon}=\varepsilon^*$), it becomes impossible for bubbles to arise (i.e., the measure of sunspot equilibria becomes zero).

In the steady state, equations (57) through (60) become

$$1=\beta(1-\mathbf{p})Q(\varepsilon^*) \tag{63}$$

$$I=(\alpha Y+\Omega q)\,F(\varepsilon^*) \tag{64}$$

$$1-\beta(1-\delta)=\beta\alpha\frac{Y}{K}\frac{Q(\varepsilon^*)}{\varepsilon^*} \tag{65}$$

$$\delta K=I\frac{\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}dF}{F(\varepsilon^*)}. \tag{66}$$

Equation (63) solves for the cutoff value ε^*. Equation (65) implies the capital-to-output ratio, $\frac{K}{Y}=\frac{\beta\alpha}{1-\beta(1-\delta)}\frac{Q}{\varepsilon^*}$. Equations (66) and (65) give the household's saving rate, $\frac{I}{Y}=\frac{\alpha\beta\delta}{1-\beta(1-\delta)}\frac{FQ}{Q-1+F}$, where $Q-1+F=\varepsilon^*\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}dF$. Equation (64) implies the asset-to-output ratio as a function of the saving rate, $\frac{\Omega q}{Y}=\frac{1}{F(\varepsilon^*)}\frac{I}{Y}-\alpha$. As in the basic model, to ensure $\frac{\Omega q}{Y}>0$, we must have $\frac{I}{Y}>\alpha F$, which implies $\delta>\left(\beta^{-1}-1+F\right)(1-\beta(1-\delta))$.

3.3 Calibration and Impulse Responses

Assume that $\varepsilon(i)$ follows the Pareto distribution, $F(\varepsilon)=1-\varepsilon^{-\theta}$, with the shape parameter $\theta=1.5$ and the support $(1,\infty)$.[15] With this distribution, we have $Q=\frac{\theta}{1+\theta}\varepsilon^*+\frac{1}{1+\theta}\varepsilon^{*-\theta}$. We normalize the steady-state values $z=1$ and $A=1$ and calibrate the structural parameters of the model as follows: The time period is a quarter, the capital's income share $\alpha=0.4$, the time-discounting factor $\beta=0.99$, the capital depreciation rate $\delta=0.025$, and the inverse elasticity of labor supply $\gamma_n=0$ (indivisible labor).

Under these parameter values, we can show that the aggregate investment-to-output ratio ($\frac{I}{Y}$) increases with the quantity of tulip bubbles. Namely, the investment-to-output ratio is an increasing function of the steady-state probability of bubbles to burst ($\bar{\mathbf{p}}$) or the asset-to-output ratio ($\frac{\Omega}{Y}$). For example, when $\bar{\mathbf{p}}=\mathbf{0.1}$, we have $\frac{\Omega q}{Y}=0.147$ and $\frac{I}{Y}=0.253$, and when $\bar{\mathbf{p}}=\mathbf{0.22}$, we have $\frac{\Omega q}{Y}=0.0034$ and $\frac{I}{Y}=0.249$.

The driving processes of the model are assumed to follow AR(1) processes,

$$\ln\mathbf{p}_t=\rho_{\mathbf{p}}\ln\mathbf{p}_{t-1}+(1-\rho_{\mathbf{p}})\ln\bar{\mathbf{p}}+\varepsilon_{\mathbf{p}t} \tag{67}$$

$$\ln A_t=\rho_A\ln A_{t-1}+\varepsilon_{At}, \tag{68}$$

where the steady-state probability for bubbles to burst is set to $\bar{\mathbf{p}}=0.1$. We set $\rho_{\mathbf{p}}=\rho_A=0.9$. Figure 2 plots the impulse responses of the model economy to a 10% increase in the probability for bubbles to burst ($\varepsilon_{\mathbf{p}t}$) and compares them with those to a 1% decrease in total productivity ε_{At} (figure 3). A positive shock to $\varepsilon_{\mathbf{p}}$ is akin to a financial crisis because it implies a higher systemic financial risk.

The solid lines in Figure 2 shows that a persistent increase in systemic risk (i.e., the bubbles' probability to burst $\mathbf{p}_t$) generates a prolonged recession in aggregate output (upper-left window) and a dramatic drop

[15] The results are not sensitive to the values of θ. For example, with $\theta=3$ we obtain similar results.

in asset prices (lower-left window). When the perceived rate of return to tulips decreases (or the financial risk increases), agents rationally decrease their demand for tulips in the anticipation that the demand for tulip asset will be persistently low in the future, leading to a sharp fall in asset price. Because tulip assets provide liquidity for firms, "panic" sales of tulips reduce firms' net worth and working capital, leading to declines in output, employment, and capital investment. The decline of output is U-shaped because tulip assets enter firms' investment decision rules (equation (58)) directly as a stock variable—less tulip investment today reduces a firm's cash position and capital investment for many periods to come and thus propagates the impact of a temporary risk shock over time. Thus, a persistent risk shock will tend to induce an AR(2) dynamic structure in output and other variables. Such a hump-shaped output dynamics suggest that asset prices lead output over the business cycle—this phenomenon provides a key litmus test of business cycle models (see Cogley and Nason, 1995). Our model passes this test with flying colors in this dimension. That the sharp decline in asset prices is caused by firms' reduction of asset demand because of the anticipated fall in asset returns in the future.

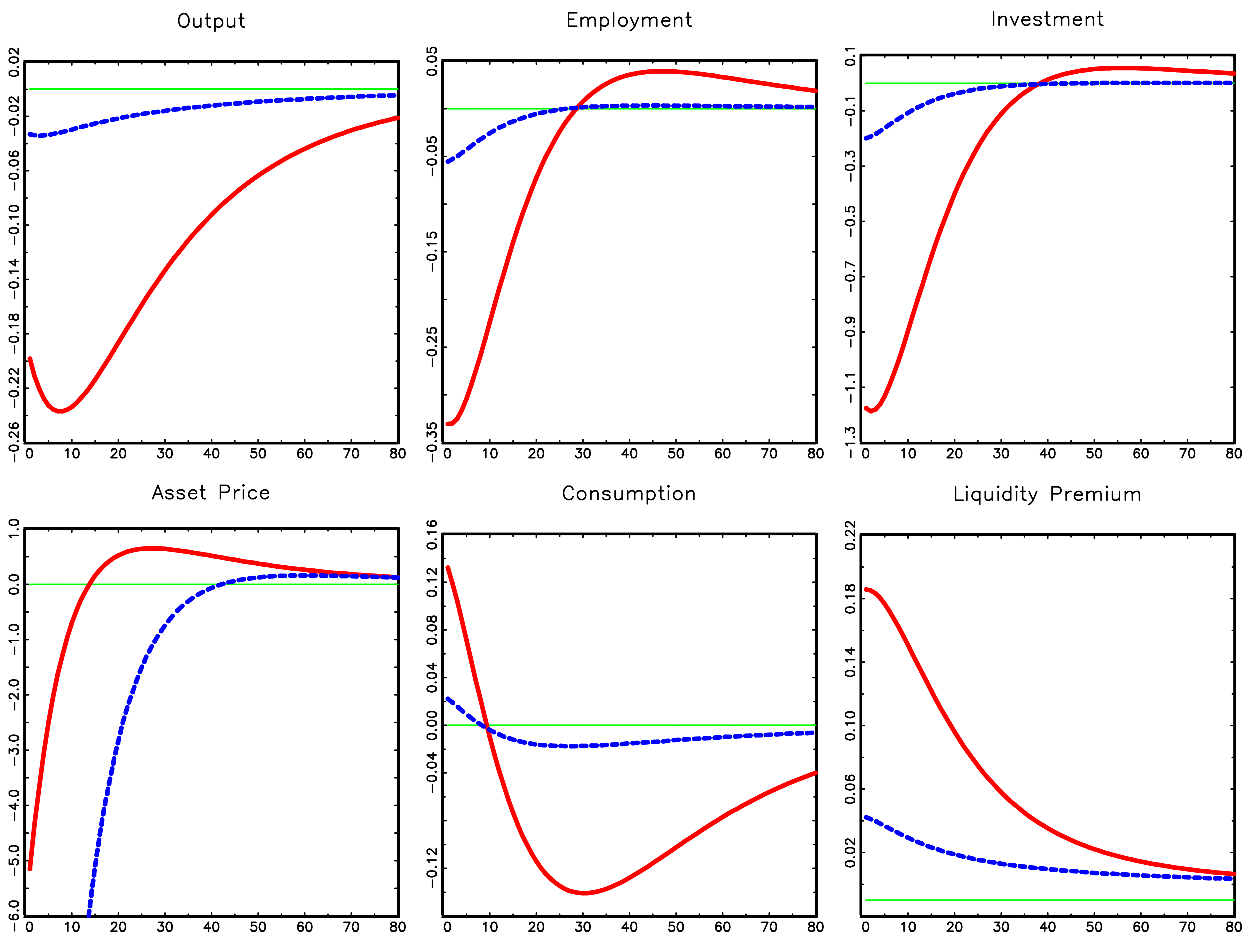

Figure 2. Impulse Responses to Risk Shock (——: $\bar{\mathbf{p}} = 0.1$, - - - -: $\bar{\mathbf{p}} = 0.22$).

More importantly, asset prices are far more volatile than the fundamentals. For example, the initial drop in asset prices is more than 25 times larger than that in output, resembling a typical stock market crash.[16] The figure suggests that a 50% fall in asset prices can cause more than 2% decrease in output. This

[16] The dynamic movements in the number of active firms who make capital-investment is quantitatively important for our

magnitude is similar to the recent U.S. stock-market experience during the subprime mortgage crisis.[17] In the meantime, aggregate employment (upper-middle window) and investment (upper-right window) also decrease sharply after the shock, with investment dropping nearly six times more than output on impact. Aggregate consumption (lower-middle window) significantly lags output. The reason that consumption responds to the shock positively in the initial period is because of the higher dividend income paid to households after a sharp reduction in fixed investment. However, consumption eventually decreases because of the persistently lower aggregate income.

Liquidity premium (Q_t) is countercyclical in our model (see the lower-right window in figure 2). Equation (57) indicates that the expected asset return is determined by $(1-\mathbf{p})\,Q$. Thus, a higher probability for asset bubbles to burst requires a higher liquidity premium to induce firms to hold bubbles in equilibrium. However, the increase in the liquidity premium will be smaller in absolute magnitude than the increase in $\mathbf{p}$ because the anticipated positive growth in asset price ($E_t \frac{q_{t+1}}{q_t}$) lowers the required liquidity premium along the transitional path. Because $Q(\varepsilon_t^*)$ is a linearly increasing function of the cutoff in the log-linear economy, the lower-right window also suggests that the number of firms investing rises during a downturn (because the cutoff ε_t^* is higher). This result is anticipated by Proposition 3. However, the number of investing firms declines during a recession if the downturn is caused by a TFP shock rather than a financial shock (see the analysis below).

To show that the existence of tulip bubbles amplify the business cycle, we also plot in figure 2 the impulse responses of the same set of variables to a systemic risk shock (of the same magnitude) when the calibrated steady-state quantity of tulips is $\bar{\mathbf{p}} = 0.22$ instead of $\bar{\mathbf{p}} = 0.1$ (namely, the steady-state probability for bubbles to burst is higher and the steady-state asset-to-output ratio $\frac{\Omega q}{Y}$ is lower). The results are represented in figure 2 as dashed lines. For example, the top windows shows that output, employment, and capital investment become much less volatile when the steady-state quantity of bubbles is reduced. This suggests that the existence of bubbles amplify the impact of financial shocks. In particular, as the steady-state quantity of bubbles reduce to zero (i.e., as $\bar{\mathbf{p}}$ increases), systemic risk will cease to have any impact on the economy in the limit (except the asset price q_t).[18]

Proposition 3 states that the investment-to-output ratio may not be necessarily higher in a bubble steady state than a no-bubble steady state. However, the impulse response analysis in figure 2 shows that along the transitional path the investment-to-output ratio increases with the quantity of bubbles because investment is more volatile than output under $\mathbf{p}_t$ shocks.

The solid lines in figure 3 shows that an adverse shock to aggregate TFP also generates a fall in asset prices (lower-left window in figure 3) that is twice as large as the fall in output (upper-left window). The asset price falls because firms reduce tulip demand when their investment demand declines after a reduction in TFP. However, the impulse response of output is not hump-shaped and the relative asset-price volatility is not big enough to match the U.S. data. In addition, the liquidity premium drops along with the asset price (lower-right window). The reason is that the supply of tulips has not changed but the demand for tulips has declined. This excess supply of tulips leads to an immediate drop in asset price but also an anticipated rise in future asset price (namely, $E_t\left[q_{t+1} - q_t\right] > 0$). Hence, the equilibrium liquidity premium will decline accordingly. Therefore, our analysis suggests that to understand the excessive asset market volatilities and

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results. For example, if we allow binary distribution for the idiosyncratic shocks as in Kiyotaki and Moore (2008), the relative volatility of asset price would reduce by more than 20%.

[17] For example, between 2008 to 2009, the annual real GDP in the United States declined by 2.63% and the S&P 500 price index dropped by 44% (this drop in stock price is more than 50% from the peak in 2007).

[18] Unlike the other economic variables, asset price q_t is more volatile relative to its steady-state value when there are less amount of tulips (see the dashed line in the lower-left panel in figure 2).

movements of liquidity premium over the business cycle, shocks to systemic financial risk are more important than shocks to TFP.

As a comparison, we also show that tulip bubbles amplify the effects of TFP shocks, albeit not as significantly as the case of financial risk shocks (see the dashed lines in figure 3). For example, when the steady-state asset-to-output ratio is low ($\bar{\mathbf{p}} = 0.22$ and $\frac{\Omega q}{Y} = 0.0034$), the declines of aggregate output, employment, and investment (dashed lines in the upper windows of figure 3) are less than in the case where the steady-state asset-to-output ratio is high (solid lines in the upper windows where $\bar{\mathbf{p}} = 0.1$ and $\frac{\Omega q}{Y} = 0.147$). Under the current parameter configurations, the maximum value of $\bar{\mathbf{p}}$ to permit a bubble steady state is $\bar{\mathbf{p}} = 0.225$, in which case the impulse responses are virtually indistinguishable to the dashed lines in figure 3.

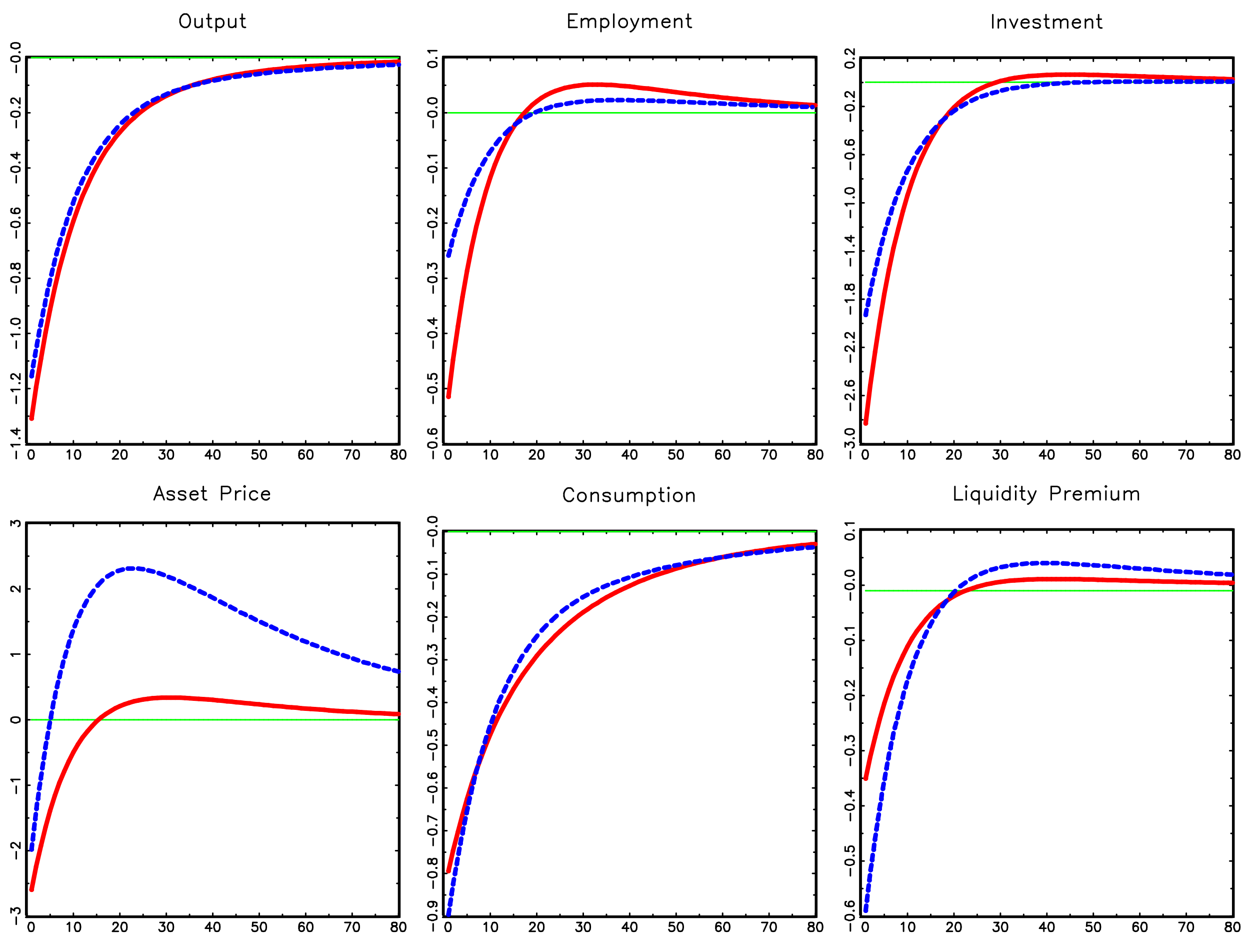

Figure 3. Impulse responses to TFP shock (——: $\bar{\mathbf{p}} = 0.1$, - - - -: $\bar{\mathbf{p}} = 0.22$).

To test whether the model has the potential to match the U.S. time-series data quantitatively, we calibrate the driving process of technology shocks $\{A_t\}$ using the Solow residual and choose the parameters $\{\rho_{\mathbf{p}}, \sigma_{\varepsilon \mathbf{p}}\}$ for the driving process $\{\mathbf{p}_t\}$ so that the model-predicted asset price volatility (σ_q) relative to output volatility (σ_Y) matches the empirical counterpart ($\frac{\sigma_q}{\sigma_Y}$) of the U.S. economy. More specifically, we follow King and Rebelo (1999) by setting $\rho_A = 0.98$ and $\sigma_{\varepsilon A} = 0.0072$ for technology shocks. Since many asset prices share similar business-cycle properties with the stock prices, we use real S&P 500 price index (deflated by GDP deflator) from the U.S. economy (1947:1–2009:1) as a proxy for the empirical counterpart of asset prices q_t.[19] The estimated standard deviation of S&P 500 price index is $\sigma_{SP} = 0.099$, which is 5.93 times the standard deviation of real GDP ($\sigma_y = 0.017$). Similar to King and Rebelo (1999), we apply the Hodrick-Prescott filter to both the U.S. data and the model generated time series prior to moment estimation. That is, we apply the HP filter on the logged series and estimate the second moments. With the specified driving process for $\{A_t, \mathbf{p}_t\}$, we find that setting $\rho_{\mathbf{p}} = 0.9$ and $\sigma_{\varepsilon\mathbf{p}} = 12 \times 0.0072 = 0.0864$ in our model would generate a ratio of $\frac{\sigma_q}{\sigma_Y} = 6.0$ in the model. Thus, our model is able to exactly match the relative volatility of asset prices in the U.S. data by properly choosing the two parameters of the driving process $\{\mathbf{p}_t\}$. The calibrated parameter values are summarized in table 1.

Table 1. Parameter Values

β	α	δ	γ_n	θ	$\bar{\mathbf{p}}$	ρ_A	$\sigma_{\varepsilon A}$	$\rho_{\mathbf{p}}$	$\sigma_{\varepsilon\mathbf{p}}$
0.99	0.4	0.025	0	1.5	0.1	0.98	0.0072	0.9	0.0864

To check whether the calibrated values for $\left\{\rho_{\mathbf{p}}, \sigma_{\varepsilon\mathbf{p}}\right\}$ are reasonable, we estimate an univariate AR(1) model for the HP filtered real S&P 500 price index and obtain an AR(1) coefficient of 0.85 and a standard deviation of the innovation 0.056, which are close in magnitude to the values of $\left\{\rho_{\mathbf{p}}, \sigma_{\varepsilon\mathbf{p}}\right\}$. Thus, we believe our calibrated values of $\left\{\rho_{\mathbf{p}}, \sigma_{\varepsilon\mathbf{p}}\right\}$ are empirically reasonable. Based on the calibrated values in table 1, we generate samples of time series from the model, apply the HP filter on the artificial data, and estimate the model's second moments. Table 1 reports the predicted second moments of the model and their counterparts in the data (where $\Delta q \equiv E_t q_{t+1}/q_t - 1$ denotes real asset returns).[20]

Table 2. Selected Business Cycle Moments

	standard deviation (σ_x)						std. relative to Y ($\frac{\sigma_x}{\sigma_Y}$)				
x	Y	C	I	N	q	Δq	C	I	N	q	Δq
Data	.017	.010	.050	.016	.099	0.061	.60	2.98	.98	5.93	3.67
Model	.012	.007	.031	.007	.074	0.009	.57	2.56	.55	6.0	.75
	$corr(x_t, x_{t-1})$						$corr(x, Y)$				
x	Y	C	I	N	q	Δq	C	I	N	q	Δq
Data	.83	.81	.84	.90	.80	0.39	.78	.77	.83	.46	-0.15
Model	.75	.75	.76	.77	.71	0.70	.89	.95	.88	.73	-0.47

* Y: output, C: consumption, I: investment, N: hours, q: asset price, Δq: asset returns.

According to table 2, the model's predictions are broadly consistent with the U.S. data. For example, in terms of standard deviations (top-left panel in table 2), the model is able to explain 70 percent of output fluctuations and 75 percent of stock price volatility in the data. In terms of relative volatilities with respect to output, the model predicts that consumption is about 40 percent less volatile, investment about 2.6 times more volatile, and asset price 6 times more volatile than output (this value is calibrated); these predictions are broadly consistent with the U.S. economy. In the data, the correlation between stock prices and output at the business cycle frequencies is 0.46; this value is 0.73 in the model, qualitatively matching the data. The model also can generate strong autocorrelations in output, consumption, investment, labor, and asset prices

[19] Variables for the U.S. economy are defined as follows: y denotes real GDP, c real non-durable consumption, i real fixed investment, , and n total hours worked for private firm employees.

[20] The data are quarterly and include real GDP (y), nondurable goods consumption (c), total fixed investment (i), total private employment by establishment survey, and the S&P 500 price index normalized by the GDP deflator. The sample covers the period 1947:1–2009:1.

that are close to the data. As in the data, the model also predicts a negative correlation between expected asset returns and output (see the last column in the lower-right panel). The gap between model and data is most significant in the relative volatility of asset returns and employment with respect to output: The model explains less than 15% of the volatility of asset returns and less than half of the volatility in employment (even with indivisible labor).

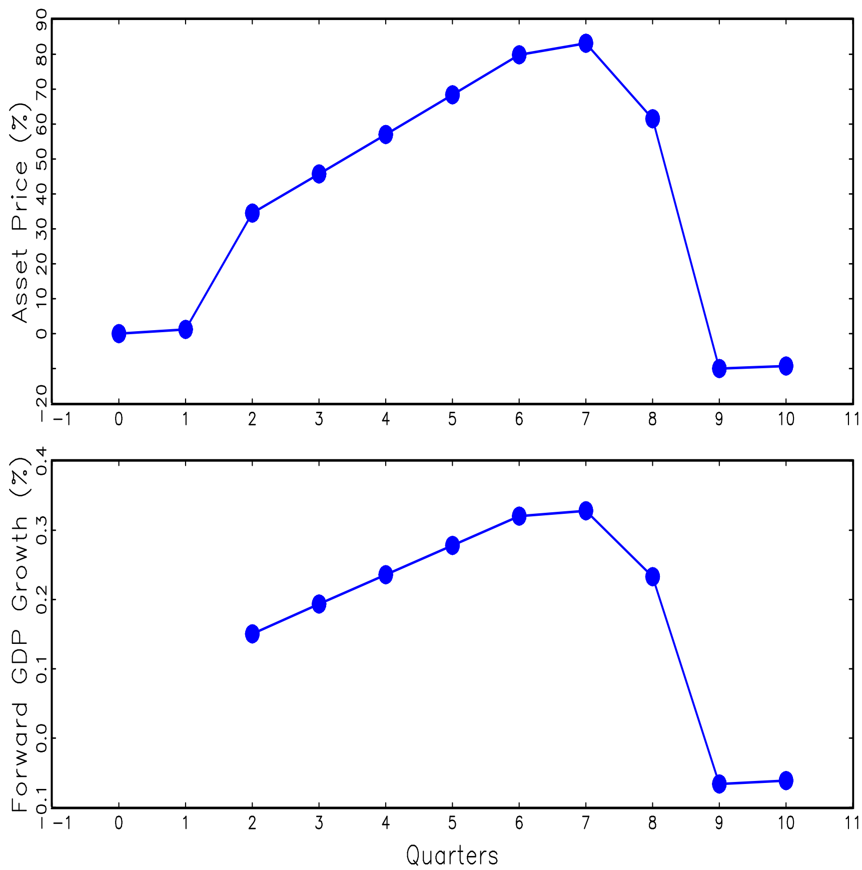

Figure 3. Simulated Tulip Bubble.

We can simulate a tulip bubble as shown in Figure 1 using the model. For example, assuming the time period to be a quarter and letting the probability of bubble to burst follow a moving average process,

$$\mathbf{p}_t = \bar{\mathbf{p}} + \sum_{j=0}^{T-1} \alpha_j \mathbf{v}_{t-j}, \tag{69}$$

where $\mathbf{v}$ represents zero mean i.i.d. innovations. Suppose $T = 9$, $\bar{\mathbf{p}} = 0.1$, and the probability weight vector $\boldsymbol{\alpha} = \frac{1}{100}[1.25, 3, 1, 1, 1, 1, 0.25, -2, -6.5]$. The simulated tulip bubble is graphed in Figure 3 (top panel). The larger the value of $\bar{\mathbf{p}}$, the larger the bubble will be in terms of magnitude. The vector $\boldsymbol{\alpha}$ has zero mean and determines the shape of the bubble. The intuition behind the values of $\boldsymbol{\alpha}$ is as follows: Because agents are forward looking, they react to good financial news by buying tulips now when they perceive that the probability of the bubbles to burst will be lower several periods from now. Thus, tulip prices would increase immediately. To prevent a big jump in the current tulip prices, there must be enough bad news today so that investors are cautious in entering the tulip market. This is why $\boldsymbol{\alpha}$ takes positive values initially so that the bubble only grows slowly and gradually. Because of the internal propagation mechanism to transmit current shocks into the future, the asset price overshoots its steady-state value from above after the market crash and does not return to the steady state immediately for a while. The lower panel in figure 3 shows the one-period ahead forecast of output growth. The figure shows that output growth follows a similar bubbly path to the asset price except with a magnitude that is a couple of orders smaller than asset prices.

4 Conclusion

This paper provides an infinite-horizon DSGE model with incomplete financial markets to explain asset bubbles and asset price volatility over the business cycle. It characterizes conditions under which bubbles with market values exceeding their fundamental values may arise. It is shown that rational agents are willing to invest in such bubbles despite their positive probability to burst and that changes in the perceived systemic risk in the asset market can trigger boom-bust cycles and asset price collapse. Calibration exercises confirm that the model has the potential to quantitatively explain the U.S. business cycle, especially the hump-shaped output dynamics) and asset price volatility.

However, as point out by Santos and Woodford (1997), bubbles in our model are fragile in the sense that if there exist interest-bearing assets that are as liquid as the tulips, there will be no bubbles in our model. As potential research topics in the future, it would be interesting to consider welfare analysis and optimal policies in our bubble economy as in Kiyotaki and Moore (2008) and Kocherlakota (2009). Another interesting avenue of research is to study bubbles with nonstationary prices. We leave these issues to future research.

Appendix I: Proof of Proposition 1

Proof. Let $\{\mu(i), \phi_t(i), \pi(i), \lambda(i)\}$ denote the Lagrangian multipliers of constraints (8) through (11), respectively; the first-order conditions for $\{i_t(i), k_{t+1}(i), h_{t+1}(i)\}$ are given, respectively, by

$$1 + \mu_t(i) = \frac{\lambda_t(i)}{\varepsilon_t(i)} + \pi_t(i) \tag{70}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ [1 + \mu_{t+1}(i)] R_{t+1} + (1-\delta)\lambda_{t+1}(i) \right\} \tag{71}$$

$$[1 + \mu_t(i)] (q_t + \zeta) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1} \left[1 + \mu_{t+1}(i) \right] \right\} + \phi_t(i), \tag{72}$$

plus the complementary slackness conditions,

$$\pi_t(i) i_t(i) = 0 \tag{73}$$

$$\phi_t(i) h_{t+1}(i) = 0 \tag{74}$$

$$[1 + \mu_t(i)] [R_t k_t(i) - i_t(i) + q_t h_t(i) - (q_t + \zeta) h_{t+1}(i)] = 0. \tag{75}$$

Notice that equation (71) implies that the value of $\lambda_t(i)$ is the same across firms because $\varepsilon(i)$ is i.i.d. and is orthogonal to aggregate shocks.

We derive the optimal decision rules of firms by a quess-and-varify strategy. Given that $\varepsilon_t(i)$ is *i.i.d.*, we conjecture that the Lagrangian multipliers, $\{\mu_t(i), \phi_t(i), \pi_t(i)\}$, depend only on idiosyncratic shocks and the aggregate states in period t. Thus, by the orthogonality assumption between aggregate and idiosyncratic shocks and the law of iterated expectations, the first-order conditions (70) to (72) can be rewritten as

$$1 + \mu_t(i) = \frac{\lambda_t(i)}{\varepsilon_t(i)} + \pi_t(i) \tag{76}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ [1 + \bar{\mu}_{t+1}] R_{t+1} + (1-\delta)\bar{\lambda}_{t+1} \right\} \tag{77}$$

$$[1 + \mu_t(i)] (q_t + \zeta) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1} \left[1 + \bar{\mu}_{t+1} \right] \right\} + \phi_t(i), \tag{78}$$

where $\bar{\mu}_t \equiv \int \mu_t(\varepsilon) dF(\varepsilon)$, $\bar{\phi}_t \equiv \int \phi_t(\varepsilon) dF(\varepsilon)$, and $\bar{\lambda}_t = \int \lambda_t(\varepsilon) dF(\varepsilon)$ denote expected values.

The decision rules at the firm level are characterized by a cutoff strategy. Notice that equation (77) implies that $\lambda_t(i) = \lambda_t$ is independent of i. Define the cutoff ε_t^* such that $\pi_t(\varepsilon_t^*) = \mu_t(\varepsilon_t^*) = 0$; then by equation (76), we have $\varepsilon_t^* = \lambda_t$. Thus, by equation (77), we have

$$\varepsilon_t^* \equiv \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ [1 + \bar{\mu}_{t+1}] R_{t+1} + (1-\delta)\bar{\lambda}_{t+1} \right\}, \tag{79}$$

which determines the cutoff. The intuition for the cutoff to equal λ_t is that λ_t is the marginal value of installing one unit of capital, but to generate one unit of capital a firm needs to invest $\varepsilon_t(i)$ units of goods. Thus, investment is profitable if and only if $\varepsilon_t(i) \le \lambda_t$. Hence, λ_t should be the cutoff ε_t^*. Also define $\phi_t^* = \phi_t(\varepsilon_t^*)$, then according to (78), we have

$$(q_t + \zeta) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1} \left[1 + \bar{\mu}_{t+1} \right] \right\} + \phi_t^*. \tag{80}$$

Now, consider two possibilities:

Case A: $\varepsilon_t(i) \leq \varepsilon_t^*$. In this case, the cost of capital investment is low. Suppose $i_t(i) > 0$; accordingly we have $\pi_t(i) = 0$. Equations (76) and (77) imply

$$\varepsilon_t(i)\left[1+\mu_t(i)\right] = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{[1+\bar{\mu}_{t+1}]R_{t+1} + (1-\delta)\bar{\lambda}_{t+1}\right\}. \tag{81}$$

Given that $\mu_t(i) \geq 0$, we must have $\varepsilon_t(i) \leq \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{[1+\mu_{t+1}(i)]R_{t+1} + (1-\delta)\lambda_{t+1}(i)\right\}$, which is precisely the cutoff defined in equation (79). Equation (76) then becomes

$$1+\mu_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)}. \tag{82}$$

Hence, whenever $\varepsilon_t(i) < \varepsilon_t^*$, we must have $\mu_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)} - 1 > 0$ and $d_t(i) = 0$. Equation (78) becomes

$$\frac{\varepsilon_t^*}{\varepsilon_t(i)}(q_t+\zeta) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{q_{t+1}\left[1+\bar{\mu}_{t+1}\right]\right\} + \phi_t(i). \tag{83}$$

Given the definition of ϕ_t^* in equation (80), equation (83) implies $\phi_t(i) > \phi_t^*$ whenever $\varepsilon_t(i) < \varepsilon_t^*$. Given that $\phi_t^* \geq 0$ (because $\phi_t(i) \geq 0$ for all $\varepsilon_t(i)$), the fact that $\phi_t(i) > \phi_t^*$ under Case A yields

$$\phi_t(i) > 0. \tag{84}$$

That is, if $\varepsilon_t(i) < \varepsilon_t^*$, we must have

$$h_{t+1}(i) = 0 \tag{85}$$

and

$$i_t(i) = R_t k_t(i) + q_t h_t(i). \tag{86}$$

This suggests that firms opt to liquidate all financial assets to maximize investment in fixed capital when the cost of fixed investment is low.

Case B: $\varepsilon_t(i) > \varepsilon_t^*$. In this case, the cost of investing in fixed capital is high. Suppose $d_t(i) > 0$ and $\mu_t(i) = 0$. Then equations (76) and (77) and the definition of the cutoff ε_t^* imply $\pi_t(i) = 1 - \frac{\varepsilon_t^*}{\varepsilon_t(i)} > 0$. Hence, we have $i_t(i) = 0$. In such a case, firms opt not to invest in fixed capital and instead pay shareholders a positive dividend. Given that $\mu_t(i) = 0$, equation (78) implies $\phi_t(i) = \phi_t^* \geq 0$. That is, the Lagrangian multiplier $\phi(i)$ is the same across firms under Case B because ϕ_t^* is independent of i. However, depending on the liquidation value of tulips in the next period, there are two possible choices (outcomes) for tulip investment under Case B: (B1) $\int_0^1 h_{t+1}(i)di > 0$ and (B2) $\int_0^1 h_{t+1}(i)di = 0$. The first outcome (B1) implies a positive aggregate demand for tulips (i.e., tulips are held as a store of value in the economy) because firms expect other firms to accept tulips in the future and the liquidation value is high enough to cover storage costs, so we must have $\phi_t^* = 0$. The second outcome (B2) implies that tulips are not traded and all existing tulips are consumed (i.e., paid to households as dividends); hence, we must have $\phi_t^* \geq 0$ and $h_{t+1}(i) = 0$ for all i. Under outcome (B2), we must also have $q_t = f$.

Thus, whether a positive demand exists for tulips under Case B depends on firms' expectation of the liquidation value of tulips in the future (i.e., on whether tulips are traded in the next period). Denoting

$$\Omega_{t+1} \equiv \int_0^1 h_{t+1}(i)di \tag{87}$$

as the aggregate demand of tulips in period t, the two possible outcomes (B1 and B2) under Case B imply the equilibrium complementary slackness condition,

$$\Omega_{t+1}\phi_t^* = 0. \tag{88}$$

Combining Cases A and B, the decision rule for capital investment is given by equation (18).The rate of returns to tulips depends on the expected marginal value of liquidity (cash flow), which is greater than 1 because of the option of waiting. This option value is denoted by

$$Q(\varepsilon^*) \equiv E\left[1+\mu(i)\right] = \int \max\left\{1, \frac{\varepsilon^*}{\varepsilon(i)}\right\} dF(\varepsilon) > 1. \tag{89}$$

When the cost of capital investment is low in period t (Case A), one unit of tulips yields $\frac{\varepsilon^*}{\varepsilon(i)} > 1$ units of new capital through investment by liquidating the tulip asset at market price q_t. When the cost is high in period t (Case B), firms opt to hold on to the liquid asset as inventories and the rate of return is simply 1.

Using equations (83) and (80), the value of the Lagrangian multiplier for the nonnegativity constraint (9) is determined by

$$\phi_t(i) = \begin{cases} \left(\frac{\varepsilon^*}{\varepsilon(i)} - 1\right)(q_t + \zeta) + \phi_t^* & \varepsilon(i) \leq \varepsilon^* \\ \\ \phi_t^* & \varepsilon(i) > \varepsilon^* \end{cases}. \tag{90}$$

This suggests that the cross-firm average shadow value of relaxing the borrowing constraint (9) by purchasing one additional unit of tulips is

$$\begin{aligned} \int_0^1 \phi(i)di &= (q+\zeta)\int_{\varepsilon\leq\varepsilon^*}\left(\frac{\varepsilon^*}{\varepsilon} - 1\right)dF(\varepsilon) + \phi_t^* \\ &= (q+\zeta)\left(Q(\varepsilon^*) - 1\right) + \phi^*, \end{aligned} \tag{91}$$

which is independent of i but positively related to the tulip price q. Equation (80) becomes equation (21). ■

Appendix II: Proof of Proposition 2

Proof. Because $R_t = \alpha\frac{Y_t}{K_t}$, equation (19) can be written as $\varepsilon_t^* = \beta E_t\left\{\frac{\Lambda_{t+1}}{\Lambda_t}\varepsilon_{t+1}^*\left[\alpha\frac{Y_{t+1}}{K_{t+1}}\frac{Q(\varepsilon_{t+1}^*)}{\varepsilon_{t+1}^*} + 1 - \delta\right]\right\}$. Also, the effective aggregate investment is given by $\int_0^1 \frac{i(i)}{\varepsilon(i)}di = \omega(\varepsilon_t^*)I_t$, where the coefficient $\omega(\varepsilon_t^*) \equiv \left(\int_{\varepsilon\leq\varepsilon_t^*}\varepsilon^{-1}dF\right)/F(\varepsilon_t^*) > \frac{1}{\varepsilon_t^*}$ measures the marginal efficiency of aggregate investment. By the law of large numbers, the firm-level investment decision rule in equation (18) implies the aggregate investment function in equation (26). Equation (22) is simply the aggregate production function, equation (23) is the aggregate resource constraint derived from equation (17), equation (24) pertains to the optimal labor supply decision of the household, equation (25) is the Euler equation for optimal tulip investment based on equation (21), equation (28) expresses the law of motion of aggregate capital accumulation, and equation (29) is the complementarity condition that determines whether tulips are traded or not in general equilibrium. ■

Appendix III: Proof of Proposition 3

Proof. In steady state B, by equation (40), we have

$$1-\beta(1-\delta)=\beta\delta\frac{Q(\varepsilon_b^*)}{\varepsilon_b^*\int_{\varepsilon\leq\varepsilon_b^*}\varepsilon^{-1}dF}=\beta\delta\left[1+\frac{1-F(\varepsilon_b^*)}{\varepsilon_b^*\int_{\varepsilon\leq\varepsilon_b^*}\varepsilon^{-1}dF}\right]; \tag{92}$$

whereas in steady state A, we have $\frac{I}{Y}>\alpha F(\varepsilon^*)$ by equation (32); hence, equations (33) and (34) imply

$$1-\beta(1-\delta)=\beta\delta\alpha\frac{Y}{I}\frac{Q(\varepsilon^*)F(\varepsilon^*)}{\varepsilon^*\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}dF}<\beta\delta\left[1+\frac{1-F(\varepsilon^*)}{\varepsilon^*\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}dF}\right]. \tag{93}$$

Equations **(92)** and **(93)** then imply

$$\frac{1-F(\varepsilon^*)}{\varepsilon^*\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}dF}>\frac{1-F(\varepsilon_b^*)}{\varepsilon_b^*\int_{\varepsilon\leq\varepsilon_b^*}\varepsilon^{-1}dF}, \tag{94}$$

or $\varepsilon^*<\varepsilon_b^*$. That is, there are fewer firms investing in fixed capital in the bubble equilibrium because the optimal cutoff ε^* is lower in steady state A. The marginal efficiency of aggregate investment is given by $\omega(\varepsilon^*)\equiv\frac{\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}dF}{F(\varepsilon^*)}$ in equation (28), which is decreasing in ε^*. Also, the capital-to-output ratio is decreasing in the cutoff by equations (33) and (38)**;** thus, we have $\frac{K}{Y}>\frac{K_b}{Y_b}$.

To show that the aggregate investment to output ratio is not necessarily higher in a bubble equilibrium, we need to show that this ratio is not a monotonic function of the cutoff. Note first that the aggregate investment to output ratio in our model is strictly less than its counterpart in the frictionless economy with complete markets:

$$\begin{aligned}
\frac{I}{Y} &= \frac{I}{K}\frac{K}{Y}=\frac{\beta\alpha}{1-\beta(1-\delta)}\frac{\delta}{\omega(\varepsilon^*)}\frac{Q(\varepsilon^*)}{\varepsilon^*}\\
&= \frac{\beta\alpha}{1-\beta(1-\delta)}\frac{\int_{\varepsilon_{\min}}^{\varepsilon^*}f(\varepsilon)d\varepsilon}{\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}f(\varepsilon)d\varepsilon}\frac{\int_{\varepsilon_{\min}}^{\varepsilon^*}\frac{\varepsilon^*}{\varepsilon}f(\varepsilon)d\varepsilon+\int_{\varepsilon^*}^{\varepsilon_{\max}}f(\varepsilon)d\varepsilon}{\varepsilon^*}\\
&= \frac{\beta\alpha}{1-\beta(1-\delta)}F(\varepsilon^*)\left[1+\frac{1-F(\varepsilon^*)}{\varepsilon^*\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}f(\varepsilon)d\varepsilon}\right],
\end{aligned}$$

where $\frac{\beta\alpha}{1-\beta(1-\delta)}$ is the investment to output ratio in a frictionless economy. Notice that the coefficient term $\Delta(\varepsilon^*)\equiv F(\varepsilon^*)\left[1+\frac{1-F(\varepsilon^*)}{\varepsilon^*\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}f(\varepsilon)d\varepsilon}\right]$ is bounded above by 1 because

$$F(\varepsilon^*)\left[1+\frac{1-F(\varepsilon^*)}{\varepsilon^*\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}f(\varepsilon)d\varepsilon}\right]\leq F(\varepsilon^*)\left[1+\frac{1-F(\varepsilon^*)}{F(\varepsilon^*)}\right]=1.$$

Because $\varepsilon^*\int_{\varepsilon\leq\varepsilon^*}\varepsilon^{-1}f(\varepsilon)d\varepsilon>\int_{\varepsilon\leq\varepsilon^*}f(\varepsilon)d\varepsilon=F(\varepsilon^*)$ for any interior values $\varepsilon^*\in(\underline{\varepsilon},\bar{\varepsilon})$, we have $\Delta(\varepsilon^*)<1$ for any interior values of $\varepsilon^*\in(\underline{\varepsilon},\bar{\varepsilon})$. This suggests that the investment to output ratio is the highest in a frictionless economy. Second, notice that $\Delta(\underline{\varepsilon})=1$ (by L'Hospital's rule) and $\Delta(\bar{\varepsilon})=1$. Hence, the function $\Delta(\varepsilon^*)$ must be U-shaped with negative slope for $\varepsilon^*<\tilde{\varepsilon}$ and positive slop for $\varepsilon^*>\tilde{\varepsilon}$. Therefore, whether $\frac{\partial\Delta(\varepsilon^*)}{\partial\varepsilon^*}\lesseqgtr 0$ depending on the value of ε^*. ■

Appendix IV: Proof of Proposition 4

Proof. We prove the proposition case by case.

(i) Suppose $\zeta = 0$ and $f = 0$. Then equation (36) is clearly satisfied if $\phi^* = 0$. In such a case, $\Omega = 0$ and $q = 0$ is an equilibrium because no firm has incentives to deviate by holding tulips when the liquidation value of tulips is zero. Hence, a fundamental equilibrium with $q = f = 0$ exists. To prove the bubble equilibrium, suppose $q > 0$. Equation (31) implies $Q(\varepsilon^*) = \frac{1}{\beta}$, which solves for the cutoff value ε^* as an interior point in the support $\varepsilon \in [\underline{\varepsilon}, \varepsilon_b^*]$ because $\frac{1}{\beta} > 1$, provided ε_b^* is large enough. Given this, we have $0 < F(\varepsilon^*) < 1$. Equation (33) implies the capital-to-output ratio,

$$\frac{K}{Y} = \frac{\alpha}{1 - \beta(1-\delta)} \frac{1}{\varepsilon^*}. \tag{95}$$

This ratio in conjunction with equation (34) give the aggregate saving rate,

$$\frac{I}{Y} = \frac{\alpha\delta}{1 - \beta(1-\delta)} \frac{F}{Q - 1 + F}, \tag{96}$$

where $Q - 1 + F = \varepsilon^* \int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF$. Equation (32) then implies the asset value-to-output ratio as a function of the saving rate,

$$\frac{q}{Y} = \left[\frac{I}{Y} \frac{1}{F(\varepsilon^*)} - \alpha \right]. \tag{97}$$

To ensure $q > 0$ (i.e., $\frac{q}{Y} > 0$), we must have $\frac{I}{Y} > \alpha F(\varepsilon^*)$, which implies the following restriction on the parameters:

$$\delta > \left(\beta^{-1} - 1 + F\right) \left(1 - \beta\left(1 - \delta\right)\right). \tag{98}$$

This condition is clearly satisfied if the household is sufficiently patient (i.e., β close to 1). In this case, $Q(\varepsilon^*)$ is close to 1, ε^* is close to its lower bound $\underline{\varepsilon}$, and $F(\underline{\varepsilon})$ is close to 0. Hence, $q > 0$, $\Omega = 1$, $\phi^* = 0$, and firms have incentives to hold bubbles. This is the case analyzed by Kiyotaki and Moore (2008) and Kocherlakota (2009).

(ii) Suppose $\zeta = 0$ and $f > 0$. In this case, $q < f$ is clearly not an equilibrium because the demand for tulips will rise to infinity. So let $q \ge f$. First, if $\zeta = 0$ and $f > 0$, steady state A and steady state B cannot coexist because equation (31) and equation (36) cannot hold simultaneously because Proposition 3 shows that the cutoff in steady state A must be lower than the cutoff in steady state B: $\varepsilon^* < \varepsilon_b^*$, which implies $Q(\varepsilon^*) < Q(\varepsilon_b^*)$. But equations (31) and (36) imply $Q(\varepsilon^*) > Q(\varepsilon_b^*)$ if $\zeta = 0$ and $f > 0$. Second, if β is large enough, then steady state A is an equilibrium (i.e., $1 = \beta Q(\varepsilon^*)$ by equation (31)) because an interior solution for $\varepsilon^* \in (\varepsilon, \varepsilon_b^*)$ exists according to equation (98) in the previous analysis. That is, following similar steps in case (i), equation (97) implies that $q \ge f$ is equivalent to the following condition:

$$\frac{\alpha\delta}{1 - \beta(1-\delta)} \frac{1}{\beta^{-1} - 1 + F} - \alpha \ge \frac{f}{Y}. \tag{99}$$

There exists a unique steady-state equilibrium whenever this condition is satisfied. For example, the above condition is satisfied when $\beta \to 1$. Third, if β is small enough, then steady state B is an equilibrium because equation (40) indicates that an unique interior solution for ε_b^* exists.

(iii) Suppose $\zeta > 0$ and $f \ge 0$. In this case, $q = f$, $\Omega = 0$ and $\phi^* \ge 0$ is an equilibrium if ζ is sufficiently large, because firms do not have incentives to deviate from the fundamental equilibrium by investing in tulips if the storage cost is too high. Now consider whether $q > f$ is also a possible equilibrium. In this case, equation (31) implies $\beta Q(\varepsilon^*) = \frac{q+\zeta}{q} \equiv \bar{\kappa} > 1$. Substituting this definition of $\bar{\kappa}$ into equation (33) gives

$\frac{K}{Y} = \frac{\alpha}{1-\beta(1-\delta)} \frac{\bar{\kappa}}{\varepsilon^*}$. Equation (34) gives

$$\frac{I}{Y} = \frac{\alpha\delta}{1-\beta(1-\delta)} \frac{\bar{\kappa}F}{\beta^{-1}\kappa - 1 + F}. \tag{100}$$

Because equation (32) implies (97), the requirement $q > f$ then implies

$$\frac{\alpha\delta}{1-\beta(1-\delta)} \frac{\bar{\kappa}}{\beta^{-1}\bar{\kappa} - 1 + F} - \alpha > \frac{f}{Y}. \tag{101}$$

This condition can be easily satisfied if $\beta \to 1$ and Y is large enough (e.g., with a large value of TFP).

Notice that the left-hand side of condition (99) approaches infinity as $\beta \to 1$ (because in this case $Q(\varepsilon^*) \to 1$ and $F(\varepsilon^*) \to 0$); hence, assets with any intrinsic values can always carry bubbles as long as agents are sufficiently patient. However, given β, the larger the fundamental value of an asset, the more difficult it is for bubbles to develop because when f is too high, the benefit of using tulips as a store of value does not outweigh the marginal utility of consumption.

Similarly, case (iii) (i.e., equation 101) states that the bubble-to-fundamental value ratio, $\frac{q}{f}$, can be made arbitrarily large if β is sufficiently close to 1 and if the economy is sufficiently productive (i.e., the output level is sufficiently high due to a high TFP). Case (iii) also indicates that multiple equilibria are possible when $f > 0$ if and only if the storage cost ζ is strictly positive (i.e., $\bar{\kappa} \equiv \frac{q+\zeta}{q} > 1$) but not too large. ■

Appendix V: Proof of Proposition 5

Proof. Let $\left\{\mu(i), \phi_t^j(i), \lambda(i), \pi(i)\right\}$ denote the Lagrangian multipliers of constraints (46) through (48), respectively, the first-order conditions for $\left\{i_t(i), k_{t+1}(i), h_{t+1}^j(i)\right\}$ are similar to those in the benchmark model:

$$1 + \mu_t(i) = \frac{\lambda_t(i)}{\varepsilon_t(i)} + \pi_t(i) \tag{102}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1 + \bar{\mu}_{t+1}]R_{t+1} + (1-\delta)\bar{\lambda}_{t+1}\right\} \tag{103}$$

$$[1 + \mu_t(i)]\, q_t^j = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{q_{t+1}^j \mathbf{1}_{t+1}^j \left[1 + \bar{\mu}_{t+1}\right]\right\} + \phi_t^j(i), \tag{104}$$

plus the complementary slackness conditions,

$$\pi_t(i) i_t(i) = 0 \tag{105}$$

$$\phi_t^j(i) h_{t+1}^j(i) = 0 \quad \text{for all } j \tag{106}$$

$$[1 + \mu_t(i)] \left[R_t k_t(i) - i_t(i) + \int_{j\in\Omega_t} q_t^j h_t^j(i) \mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj - \int_{j\in\Omega_{t+1}} q_t^j h_{t+1}^j(i) dj\right] = 0. \tag{107}$$

As in the benchmark model, the decision rules at the firm level are characterized by a cutoff strategy. The following steps are analogous to those in the benchmark model. Consider two possibilities:

Case A: $\varepsilon_t(i) \le \varepsilon_t^*$. In this case, the cost of capital investment is low. Suppose $i_t(i) > 0$, then $\pi_t(i) = 0$. Equations (102) and (103) imply

$$\varepsilon_t(i) \left[1 + \mu_t(i)\right] = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1 + \bar{\mu}_{t+1}]R_{t+1} + (1-\delta)\bar{\lambda}_{t+1}\right\}. \tag{108}$$

Given that $\mu_t(i) \geq 0$, we must have $\varepsilon_t(i) \leq \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1+\bar{\mu}_{t+1}]R_{t+1} + (1-\delta)\bar{\lambda}_{t+1}\right\}$, which defines the cutoff value, ε_t^*:

$$\varepsilon_t^* \equiv \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1+\bar{\mu}_{t+1}]R_{t+1} + (1-\delta)\bar{\lambda}_{t+1}\right\}. \tag{109}$$

Equation (102) then becomes $1+\mu_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)}$. Hence, whenever $\varepsilon_t(i) < \varepsilon_t^*$, we must have $\mu_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)} - 1 > 0$ and $d_t(i) = 0$. Equation (104) becomes

$$\frac{\varepsilon_t^*}{\varepsilon_t(i)} q_t^j = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1}^j \mathbf{1}_{t+1}^j \left[1+\bar{\mu}_{t+1}\right] \right\} + \phi_t^j(i). \tag{110}$$

Defining ϕ_t^* as the cutoff value of $\phi_t^j(i)$ for firms with $\varepsilon_t(i) = \varepsilon_t^*$, equation **(111)** implies

$$q_t^j = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1}^j \mathbf{1}_{t+1}^j \left[1+\bar{\mu}_{t+1}\right] \right\} + \phi_t^* \tag{111}$$

Given that $\phi_t^* \geq 0$, the fact that $\phi_t^j(i) > \phi_t^*$ under Case A yields $\phi_t^j(i) > 0$. That is, for any $\varepsilon_t(i) < \varepsilon_t^*$, we must have $h_{t+1}^j(i) = 0$ for all $j \in \Omega_{t+1}$ and $i_t(i) = R_t k_t(i) + \int_{j\in\Omega_t} q_t^j h_t^j(i) \mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj$. This suggests that firms opt to liquidate all tulip assets to maximize investment in fixed capital when the cost of fixed investment is low.

Case B: $\varepsilon_t(i) > \varepsilon_t^*$. In this case, the cost of investing in fixed capital is high. Suppose $d_t(i) > 0$ and $\mu_t(i) = 0$. Then equations (102) and (103) and the definition of the cutoff ε^* imply $\pi_t(i) = 1 - \frac{\varepsilon_t^*}{\varepsilon_t(i)} > 0$. Hence, we have $i_t(i) = 0$. In such a case, firms opt not to invest in fixed capital and, instead, pay the shareholders positive dividends. Because the market clearing condition for each tulip j is $\int h_{t+1}^j(i)di = 1$, we must also have $h_{t+1}^j(i) > 0$ and $\phi_t^j(i) = 0$ for all $j \in \Omega_{t+1}$ under Case B. Thus, equations (104) and (111) then imply $\phi_t^* = 0$.

Combining these two cases gives the decision rule for capital investment in equation (50). The option value of liquidity is again defined by $Q(\varepsilon^*) \equiv E\left[1+\mu(i)\right] = \int \max\left\{1, \frac{\varepsilon^*}{\varepsilon(i)}\right\} dF(\varepsilon)$. Using equations (110) and (111), the Lagrangian multiplier for the nonnegativity constraint (47) is given by

$$\phi_t^j(i) = \begin{cases} \left(\frac{\varepsilon^*}{\varepsilon(i)} - 1\right) q_t^j & \varepsilon(i) \leq \varepsilon^* \\ \\ 0 & \varepsilon(i) > \varepsilon^* \end{cases}, \tag{112}$$

and the average shadow value of ϕ^j is $\int \phi^j(i)di = q^j \int_{\varepsilon \leq \varepsilon^*} \left(\frac{\varepsilon^*}{\varepsilon} - 1\right) dF(\varepsilon) = q^j\left(Q-1\right)$, which is independent of i but proportional to a tulip's price, q^j. Integrating equation (104) over i and rearranging gives equation (51). ■

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