Private Information and Sunspots in Sequential Asset Markets*

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(This version: September 7, 2014)

Abstract

We study a model where some agents have private information about risky asset returns and trade to obtain capital gains, while others acquire the risky asset and hold it to maturity, forming expectations of returns based on market prices. We show that under such a structure, in addition to fully revealing rational expectations equilibria, there exists a continuum of equilibrium prices consistent with rational expectations, where the the asset prices are subject to sunspot shocks. Such sunspot shocks can generate persistent fluctuations in asset prices that look like a random walk in an efficient market.

Keywords: The Grossman-Stiglitz paradox, Sunspots

JEL codes: D82, D83, G12,G14

1 Introduction

The efficient markets hypothesis states that prices on traded assets reflect all publicly available information. In their classic work Grossman and Stiglitz (1980) discussed a model where agents can obtain private information about asset returns and can trade on the basis of that information. If however the rational expectations equilibrium price reveals the information about the asset, and if information collection is costly, then agents have no incentive to collect the information before they observe the price and trade. But then prices no longer reflect the information about the asset, and markets are no longer efficient. Since then a large empirical and theoretical literature has explored the informational efficiency of markets under private information.[1]

We study the possibility of multiple rational expectations sunspot equilibria driven by non-fundamentals in asset markets with private information by introducing a simple time dimension

*We are grateful to referees and the editor for comments and to Federico Filippini who caught an error in an early draft. Wang acknowledges the financial support from the Research Grants Council of Hong Kong under project number 693513.

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[1] See for example Malkiel (2003).

to markets where agents trade sequentially. In our simplest benchmark model short term traders have noisy information about the return or dividend yield of the asset, but hold and trade the asset before its return is realized at maturity. The returns to short-term traders consist of capital gains. Investors, on the other hand, who may not have private information about the returns or dividend yields, but can observe past and current prices, purchase and hold the asset for its final dividend return. While we do not impose constraints on borrowing, asset holdings, or short-selling,[2] we exclude traders that have private information on dividends from holding the risky asset all the way from its inception at time 0 to its maturity when terminal dividends are paid.[3] We show that under such a market structure, in addition to equilibria where equilibrium prices fully reveal asset returns as in Grossman and Stiglitz (1980), there also exists a continuum of equilibria with prices driven by sunspot shocks. These equilibria are fully consistent with rational expectations and they are not randomizations over multiple fundamental equilibria. Furthermore the sunspot or sentiment shocks generate persistent fluctuations in the price of the risky asset that look to the econometrician like a random walk in an efficient market driven by fundamentals.[4]

Multiple equilibria also arises in the paper of Cespa and Vives (2014) in a two period model of sequential trading with short term traders, as well as noise or liquidity traders whose demands for assets are persistent, or correlated across periods. Risk-averse short-term traders limit their trading because of price uncertainty, but the persistence of asset demands by liquidity traders reduces the information about fundamentals revealed by prices. This dampens the response of investor demand since prices become less informative about fundamentals (liquidation values), and thus tends to reduce price volatility. On the other hand faced lower volatility, risk-averse short-term traders trade more aggressively based on their private information. With these effects, driven by the persistence of the demand of liquidity traders across periods, we obtain two equilibria: a high information equilibrium with low volatility and prices informative about fundamentals, and a second low information equilibrium with high volatility and prices that are less informative about fundamentals.

In the next two sections we start with a simple three period model and derive results on the fundamental and the sunspot equilibria of our model, and we discuss the intuition for our results. In section 4 we study more general information and signal structures to show that our results are robust to such generalizations. We relax the assumption that all short-term traders perfectly observe the same sunspot and allow them to observe private sunspot or sentiment signals that are

[2] Compare, for example, with Miller (1977) or with Harrison and Kreps (1978) where traders hold heterogenous beliefs about terminal returns, but where short-selling constraints rule out unbounded trades.

[3] This or similar kinds of market structures, involving short-term traders and longer term investors have been widely used, for example in Cass and Shell (1983), Allen and Gorton (1993), Allen, Morris and Shin (2006), or in Angeletos, Lorenzoni and Pavan (2010) where entrepreneurs sell their investments to traders. See our discussion in Section 9.

[4] See Section 7. For a survey of the literature on asset prices driven by sentiments see Baker and Wurgler (2007).

correlated. We show that our results in the benchmark model carry over in this case. We then also allow the long-term investors to receive private signals on the dividend and on sunspot shocks that can be correlated with the signal of short term traders. We show that the sunspot or sentiment driven equilibria are robust to this generalized information structure. In Section 5 we allow long term investors to also trade in the initial period, and to obtain private signals on dividends and sunspots, which again can be correlated with the signals of short term traders. We show that we can still have a continuum of sunspot equilibria in this context, and that our results are not driven by a market structure that excludes long-term investors from trading in period 0.

In section 6 we introduce multiple assets and show that the co-movement of asset prices in excess of co-movements in fundamentals can be explained by our sunspot equilibria. In section 7 we extend our model to multiple periods. We show that asset prices under the sunspot equilibria exhibit random walk behavior even though the asset prices are not purely driven by fundamentals.

Recently Vives (2014) has raised the issue that fully revealing rational expectation equilibrium may not be implementable: if short term traders ignore their private signal at fully revealing equilibrium prices, we are faced with the problem of how fully revealing equilibrium prices incorporating these private signals are in fact realized in the market. Since Vives' critique applies in our model, we address the implementability of our equilibria in section 8 along the lines suggested by Vives (2014), and we also consider the approach of Golosov, Lorenzoni and Tsyvinski (2014). Finally to put our results in context, in Section 9 we briefly discuss, without attempting to be comprehensive, some papers in the literature with models and results that are closely related to ours.

We should also emphasize that we have deliberately not introduced any noise traders or imperfectly observed stochastic asset supplies, often used in the literature on asset prices to prevent prices from being perfectly revealing[5]. Therefore the continuum of sunspot equilibria that we obtain in our model are not related to noise traders in any way.

2 The Model

We start with a three period benchmark model with a continuum of short-term traders and long-term investors. We index the short-term trader by j and the long-term investor by i. In period 0 there is a continuum of short-term traders of unit mass endowed with 1 unit of an asset, a Lucas tree. This tree yields a dividend D in period 2. We assume that

$$\log D = \theta. \tag{1}$$

where θ is drawn from a normal distribution with mean of $-\frac{1}{2}\sigma_\theta^2$ and variance of σ_θ^2. Each trader in period 0 is a short-term trader who receives utility in period 1 and therefore sells the asset in

[5] Vives (2014) is a notable exception, which provides a solution to the Grossman-Stiglitz Paradox without resorting to noise traders.

period 1 before D is realized in period 2. This short-term trader, maybe because he is involved in creating and structuring the asset, receives a signal s_j

$$s_j = \theta + e_j \tag{2}$$

where e_j has a normal distribution with mean of 0 and variance of σ_e^2. We assume that e_j is independent of θ. So the short-term trader j in period 0 solves

$$\max_{x_{j0}, B_{j0}} \mathbb{E}[C_{j1}|P_0, s_j] \tag{3}$$

with the budget constraints

$$P_0 x_{j0} + B_{j0} = P_0 + w. \tag{4}$$

$$C_{j1} = P_1 x_{j0} + B_{j0.} \tag{5}$$

where w is his endowment or labor income, x_{j0} is the quantity of the asset and B_{j0} is a safe bond that he carries over to period 1. We assume that there is no restriction on B_{j0}. Therefore using the budget constraint we can rewrite the $j'th$ short-term trader's problem as[6]

$$\max_{x_{j0} \in (-\infty, +\infty)} \mathbb{E}[P_0 + w + (P_1 - P_0)x_{j0}|P_0, s_j] \tag{6}$$

There is a continuum of investors of unit mass in period 1 who trade with the short-term traders. Each of them is also endowed with w and enjoys consumption in period 2 when the dividend D is realized. These investors solve a similar problem, but have no direct information about the dividend of the Lucas tree, except through the prices they observe. Hence an investor i in period 1, solves

$$\max_{x_{i1}, B_{i1}} \mathbb{E}[C_{i2}|P_0, P_1] \tag{7}$$

with the budget constraints

$$P_1 x_{i1} + B_{i1} = w \tag{8}$$

$$C_{i2} = D x_{i1} + B_{i1.} \tag{9}$$

where w is his endowment, x_{i1} is their asset purchase, and B_{i1} is his bond holdings carried over to period 2. Similarly the objective function (7) can be written as

$$\max_{x_{i1} \in (-\infty, +\infty)} \mathbb{E}[w + (D - P_1)x_{i1}|P_0, P_1], \tag{10}$$

after substituting out B_{i1} from the budget constraints.

[6] Note that we are not restricting the domains of $x_{j0} \in (-\infty, +\infty)$ and $x_{i0} \in (-\infty, +\infty)$, so in principle traders and investors may choose unbounded trades in the risky asset. This of course will be impossible in equilibrium since the asset supply $x = 1$. Alternatively we could constrain trades so $x_{j0}, x_{i0} \in (-B_l, B_h)$, $B_l, B_h > 0$, with results unaffected for $B_l, B_h \geq 1$ for example.

3 Equilibrium

An equilibrium is a pair of price $\{P_0, P_1\}$ such that x_{j0} solves problem (6) and x_{i1} solves problem (10), and markets clear. Formally we define our equilibrium concept below.

Definition 1 *An equilibrium is an individual portfolio choices $x_{j0} = x(P_0, s_j)$ for the short-term traders in period 0, $x_{i1} = y(P_0, P_1)$ for the long-term investors in period 1, and two price functions $\{P_0 = P_0(\theta), P_1 = P_1(\theta)\}$ that jointly satisfy market clearing and individual optimization,*

$$\int x_{j0} dj = 1 = \int x_{i1} di, \tag{11}$$

$$P_0 = \mathbb{E}[P_1 | P_0, s_j] \tag{12}$$

for all $s_j = \theta + e_j$, and

$$P_1 = \mathbb{E}\left[D | P_0, P_1\right], \tag{13}$$

Equation (11) gives market clearing, (12) gives the first order condition for an interior optimum for the short term trader, and equation (13), the first order condition for the long term investors, says that the price that the long term investor is willing to pay is equal to their Bayesian updating of the dividend. Under these interior first order conditions our risk-neutral agents are indifferent about the amount of the asset they carry over, so for simplicity we may assume a symmetric equilibrium with $x_{i0} = x = 1$, and $x_{i1} = x = 1$. Hence the market clearing condition (11) holds automatically. In what follows, we only need to check equations (12) and (13) to verify an equilibrium.

Proposition 1 $P_0 = P_1 = \exp(\theta)$ *is always an equilibrium.*

Proof. *The proof is straightforward. It is easy to check that both (12) and (13) are satisfied.* ■

In this case, the market price fully reveals the fundamental values. Whatever their individual signal, traders in period 0 will be happy to trade at $P_0 = \exp(\theta)$, which reveals the dividend to investors in period 1. Even though each trader j in period 0 gets a noisy private signal s_j about θ, which may be high or low, these traders act as if they ignore their signal. In maximizing their utility they only care about the price at which they can sell next period. If each short term trader believes the price in the next period depends on θ, competition in period 0 will then drive the market price exactly to $\exp(\theta)$. For a given market price, the expected payoff of holding one additional asset will be $\mathbb{E}\left\{[\exp(\theta) - P_0] | P_0, s_j\right\}$. As long as $\log P_0 \neq \theta$, traders with low signals would want to short the risky asset while other traders with high signals would want to go long on the risky asset. An equilibrium can only be reached when the market price has efficiently aggregated all private information in such a way that idiosyncratic signals cannot provide any additional profits based on private information: namely $\log P_0 = \int s_j dj = \theta$. Since price fully reveals the dividend, the long term investors will be happy to pay $\log P_1 = \theta$ in the next period.

There is however a second equilibrium where the market price reveals no information about dividends.

Proposition 2 *$P_0 = P_1 = 1$ is always an equilibrium.*

Proof. *Both (12) and (13) are satisfied. It is clear that with $P_0 = P_1 = 1$, investors in period 1 obtain no information about the dividend as the prices simply reflect the unconditional expectation of the dividends in period 2.* ■

Again in the above equilibrium, the short-term traders "optimally" ignore their private signals. If the short-term traders believe that the price in the next period is independent from θ, then their private signal s_j is no long relevant for their payoff, and these signals become irrelevant.

3.1 Sentiment-Driven Equilibria

We now assume the traders in period 0 also receive some sentiment or sunspot shock z which they believe will drive prices. We are assume that z has a standard normal distribution. We define an sentiment-driven equilibrium as follows.[7]

Definition 2 *An sentiment-driven equilibrium is given by optimal portfolio choices $x_{j0} = x(P_0, s_j, z)$ for the short-term trader in period 0, $x_{i1} = y(P_0, P_1)$ for the long-term investors in period 1, and two price functions $\{P_0 = P_0(\theta, z), P_1 = P_1(\theta, z)\}$ that jointly satisfy market clearing and individual optimization,*

$$\int x_{j0} dj = 1 = \int x_{i1} di, \tag{14}$$

$$P_0 = \mathbb{E}[P_1 | P_0, s_j, z] \tag{15}$$

for all $s_j = \theta + e_j$ and z, and

$$P_1 = \mathbb{E}\left[D | P_0, P_1\right], \tag{16}$$

[7] The equilibria may be simply defined in the context of a Bayesian game where players are the short-term traders $j \in J = [0,1]$ and long-term investors $i \in I = [0,1]$, where each player is endowed with $w > 0$. Each short term trader is also endowed with one unit of the risky asset $x = 1$. The action spaces can be taken as $x_{j0} \in [-B_l, B_h] = B$ for short-term traders and $x_{j0} \in [-B_l, B_h] = B$ for the long-term investors where $B_l, B_h > 0$, and where in the paper we take $B_l = B_h = \infty$. (Of course in equilibrium the aggregate asset supply must be $x = 1$ so unbounded trades are impossible.) The states of the world $S = \left(\theta, z, \{\varepsilon_j\}_{j \in [0,1]}\right)$ are realizations of $\left(\theta, z, \{\varepsilon_j\}_{j \in [0,1]}\right)$ according to the probability distributions. For prices P_0, P_1, payoffs for short-term traders j are defined by $P_0 + w + (P_1 - P_0)x_{j0}$ and payoffs for long-term investors i by $w + (D - P_1)x_{i1}$. Strategies for short-term traders j map (P_0, s_j, z), where $s_j = \theta + e_j$, into actions $x_{j0} \in B$ in order to optimize (6), and strategies for long-term traders map (P_0, P_1) into actions $x_{i1} \in B$ to optimize (10), both using optimal Bayesian updating in the forming of expectations. Equilibrium given by Definition 2 defines equilibrium price functions under market clearing and optimization by agents.

As in the section 3, equation (14) gives market clearing, (15) gives the first order condition for an interior optimum for the short term trader and equation (16), the first order condition for the long term investors, says that the price that the long term investor is willing to pay is equal to their conditional expectation of the dividend.

Proposition 3 *There exists an continuum of sentiment driven equilibria indexed by* $0 \leq \sigma_z^2 \leq \frac{1}{4}\sigma_\theta^2$, *with* $x_{i0} = 1 = x_{j1}$ *and the prices in two periods given by*

$$\log P_1 = \log P_0 = \phi\theta + \sigma_z z, \tag{17}$$

where

$$0 \leq \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_z^2}{\sigma_\theta^2}} \leq 1. \tag{18}$$

Proof. Note that since the prices are the same in both periods, (15) is satisfied automatically. We only need to check if equation (16) is satisfied. Taking the log of equation (16) generates:

$$\begin{aligned} \log P_1 &= \phi\theta + \sigma_z z \qquad (19) \\ &= \log \mathbb{E}\{\exp(\theta|\phi\theta + \sigma_z z)\}, \\ &= \mathbb{E}\left[\theta|\phi\theta + \sigma_z z\right] + \frac{1}{2} var(\theta|\phi\theta + \sigma_z z) \\ &= -\frac{1}{2}\sigma_\theta^2 + \frac{\phi\sigma_\theta^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\left[\phi\theta + \sigma_z z + \frac{\phi}{2}\sigma_\theta^2\right] \\ &\quad + \frac{1}{2}\left[\sigma_\theta^2 - \frac{\left(\phi\sigma_\theta^2\right)^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\right]. \end{aligned}$$

which follows from the property of the normal distribution. Comparing terms, coefficients of $\phi\theta + \sigma_z z$ yields

$$\frac{\phi\sigma_\theta^2}{\phi^2\sigma_\theta^2 + \sigma_z^2} = 1. \tag{20}$$

Solving equation (20) yields the expression of ϕ in equation (18). ■

In this case traders in period 0 get a common sunspot shock z. The investors in period 1, in forming their expectation of D conditional on the prices, believe prices are affected by the sunspot z, as in equation (19). However now they have a signal extraction problem in distinguishing θ from z. Their first order conditions will be satisfied in equilibrium provided the variance of z lies within the interval given in Proposition 3, generating a continuum of sunspot equilibria indexed by σ_z^2. For example, a low z will induce pessimistic expectations for the period 0 trader, who will pay a low price for the asset and expect a low price next period. The investor in period 1 will observe the period 0 price and infer that in part, this must be due to a low dividend yield, which will lead him to also pay a low price in period 1, thus confirming the expectations of the period 0 trader. Of course for this to be possible for every realization of the sunspot z, the variance of z that enters the signal extraction problem of the investor in period 1 must lie in the interval given in the above Proposition.

We can understand the intuition behind the multiplicity by analogy to the Keynesian Beauty Contest put in the context informational asymmetries and correlated signals. Note that the multiplicity of equilibria in our model does not hinge on the precision of signal s_j. We may, for simplicity, assume that $s_j = \theta$, so the short term investors are assumed to know the dividend for sure. The price revealing equilibrium in this case would be $\log P_0 = \log P_1 = \theta$. As in the Keynesian Beauty Contest however, even though short-term traders' own view of the true value of stock (the dividend) is equal to θ, this does not matter to them. If the other short term traders think the price is different from θ, a short term trader will still be willing to accept such a price as long as he can sell it in the next period at the same price. So any price can support an equilibrium from the short term trader's point of view. However, a price can only be an equilibrium price if the long term investors are willing to trade the asset at such a price. In order for a rational expectation sunspot equilibrium to exist, the price has to reflect the fundamental dividend value θ with noise in such a way that the Bayesian updating of the fundamental dividend value θ exactly equals the market price. This gives a restriction on the coefficient ϕ and the variance of noise in the price rule (19). The role of sunspots then is to correlate and coordinate the behavior of short term traders so the investors, knowing the variance of sunspots and fundamentals, can optimally update their expectation of the dividends that they will collect in equilibrium.

In our three-period model the assumption that short term traders can not participate in period 2 trades is important. If the short term traders are allowed to trade in period 2, then the multiplicity of equilibrium will disappear. In such a case if the price $\log P_0 = \log P_1 < \theta$, then the short term trader will opt to go long on the asset in period 1. The purchase of each additional unit of the risky asset will increase his utility by $\exp(\theta) - P_1 > 0$. Competition will then bring the price to $\log P_0 = \log P_1 = \theta$. Likewise any price (in logs) that is above θ will induce the short term trader to short the asset, forcing $\log P_0 = \log P_1 = \theta$ in equilibrium.

For the same reason in a multi-period context with periods $t = 0, ...T + 1$, such as in Section 7, for sunspots to exist we have to rule out traders that can hold risky assets all the way from period 0 to maturity at $T + 1$: If the market price at any t differs from such traders' expectations of the terminal dividend θ, they will be able to arbitrage the difference by buying or short-selling the asset, unless we explicitly introduce borrowing or short-selling constraints to prevent arbitrage.

In the following sections we will relax the informational assumptions of our model and generalize it to multiple assets and periods.

4 Alternative Information Structures

We examine the robustness of our results to alternative information structures. We first relax the assumption that all short-term traders perfectly observe the same sunspot. Instead, we assume that they observe private sunspots or sentiments that are correlated. Thus their sentiments are heterogenous but correlated. We show that our results in the benchmark model carry over in this case. We then also allow the investors to receive some private signal on the dividend and on sunspot shocks. We show that the sunspot or sentiment driven equilibria are robust to this generalized information structure. For expositional convenience, we denote Ω_0 and Ω_1 as the information sets of a particular short-term trader and the investor, respectively. The equilibrium conditions can then be written as

$$P_0 = \mathbb{E}[P_1|\Omega_0], \tag{21}$$

and

$$P_1 = \mathbb{E}\left[D|\Omega_1\right]. \tag{22}$$

We can now proceed to study alternative of the information sets Ω_0 and Ω_1.

4.1 Heterogenous but Correlated Sentiments

If each short-term trader receives a noisy sentiment or sunspot shock z, then the information set Ω_0 for a particular trader becomes $\Omega_0 = \{P_0, \theta + e_j, z + \varepsilon_j\}$, where ε_j are drawn from a normal distribution with mean of 0 and variance of σ_ε^2 and $cov(e_j, \varepsilon_j) = 0$. Note again the the sentiment or sunspot shocks are correlated across traders due to the common component z. Furthermore $\Omega_1 = \{P_0, P_1\}$ is the same as in our benchmark model of the previous section. In this case, equilibrium prices still take the form described in equation (17). Namely there exist an continuum of sunspot equilibria indexed by the sunspot's variance as in Proposition 3. It is easy to check that (21) holds for any realization of e_j and ε_j, hence the short-term trader's first order conditions hold. Since the information set Ω_1 is the same as in the benchmark model, equation (22) will be automatically satisfied. The market "efficiently" washes out the noise ε_j. The intuition is similar to the fully revealing equilibrium in the benchmark model. The market clearing condition must make the agents ignore their private signal. Otherwise agents with a high realization of $\theta + e_j$ or $z + \varepsilon_j$ would go long on the assets while agents with low signals would keep shorting them, destroying the market equilibrium. An equilibrium can be reached only if no agent has an incentive either to short the asset or go long on it based on his own private information. In equilibrium the prices must include all private information, so that given the market price no agent can gain any informational advantage based on his own signals.

4.2 The Investors and Market Signals on the Dividend and on Sunspots

We first relax assumption that only the short-term investor receives information about the dividend through a private signal. We allow both the short term trader and the investor to receive private information on the dividend θ. We change the information set to $\Omega_0 = \{P_0, \theta_0 + e_j, z + \varepsilon_j\}$ and $\Omega_1 = \{P_0, P_1, \theta_1 + v_i\}$. Here $s_{j0} = \theta_0 + e_j$ is the private signal on the dividend received by a trader j in the first period, and $s_{i1} = \theta_1 + v_i$ is the signal of the investor i in the second period. We assume that $cov(\theta_0, \theta) > 0$ and $cov(\theta_1, \theta) > 0$, but $cov(\theta_0, \theta_1) = 0$. For example, $\theta = \alpha\theta_0 + (1-\alpha)\theta_1$, with $0 < \alpha < 1$, but $cov(\theta_0, \theta_1) = 0$ satisfies these assumptions. Without loss of generality we assume $\theta = \theta_0 + \theta_1$. In addition, we assume that θ_0 (θ_1) are drawn from normal distribution with mean of $-\frac{1}{2}\sigma^2_{\theta_0}$ ($-\frac{1}{2}\sigma^2_{\theta_1}$) and variance of $\sigma^2_{\theta_0}$ ($\sigma^2_{\theta_1}$) and v_i is normally distributed with mean of 0 and variance of σ^2_v. The equilibrium conditions are again given by (21) and (22). Proposition 4 specifies the equilibrium prices with such an information structure[8].

Proposition 4 *The exists an continuum of sunspot equilibria indexed by* $0 \le \sigma^2_z \le \frac{1}{4}\sigma^2_{\theta_0}$, *where equilibrium prices are given by*

$$\log P_0 = \phi\theta_0 + \sigma_z z, \tag{23}$$

$$\log P_1 = \phi\theta_0 + \sigma_z z + \theta_1, \tag{24}$$

where ϕ *is given by*

$$0 \le \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma^2_z}{\sigma^2_{\theta_0}}} \le 1. \tag{25}$$

Proof. *The proof is similar to Proposition 3. Plugging the expression of* $\log P_0$ *and* $\log P_1$ *into (21), we obtain*

$$\phi\theta_0 + \sigma_z z = \phi\theta_0 + \sigma_z z + \mathbb{E}(\theta_1|\Omega_0) + \frac{1}{2}var(\theta_1|\Omega_0). \tag{26}$$

Since θ_1 *is independent of* Ω_0, *we have* $\mathbb{E}(\theta_1|\Omega_0) + \frac{1}{2}var(\theta_1|\Omega_0) = -\frac{1}{2}\sigma^2_{\theta_1} + \frac{1}{2}\sigma^2_{\theta_1} = 0$. *Therefore equation (21) is satisfied for any trader* j *in period* 0. *We now turn to equation (22). Notice that by studying the prices in the two periods, the investor can now learn* θ_1 *with certainty. Hence we*

[8] In what follows we assume that correlation between two random variables, if not explicitly specified, is zero.

have

$$\log P_1 = \mathbb{E}(\theta|\Omega_1) + \frac{1}{2}var(\theta|\Omega_1)$$
$$= \theta_1 + \mathbb{E}(\theta_0|\phi\theta_0 + \sigma_z z) + \frac{1}{2}var(\theta_0|\phi\theta_0 + \sigma_z z)$$
$$= \theta_1 + \frac{\phi\sigma_{\theta_0}^2}{\phi^2\sigma_{\theta_0}^2 + \sigma_z^2}\left[\phi\theta_0 + \sigma_z z\right], \quad (27)$$

Since ϕ is given by (25), we have $\phi\sigma_{\theta_0}^2 = \phi^2\sigma_{\theta_0}^2 + \sigma_z^2$. Hence equation (22) holds as well. ■

In this case, the market efficiently aggregates the private information of long term investors. The price in period 1 has to incorporate all the private information of the long term investors, otherwise some investors with high/low signals on the underlying dividend would attempt to profit by shorting/longing the assets. However, the period 1 price only partially incorporates the private information of the short term traders regarding the underlying dividend. These short-term traders benefit only from potential capital gains, and are risk neutral. As long as the expected return is equal to the risk free rate, they will not care whether the prices are driven by sunspots or by fundamentals and sunspot equilibria will continue to exist.

We now further generalize the information structure by allowing the investors to also receive signals on the sunspots and as well as the dividends observed by the short term traders. We assume that $\theta = \theta_0 + d + \theta_1$ and $z = z_0 + \xi + z_1$. The information set of the short term trader is $\Omega_0 = \{P_0, \theta_0 + d + e_j,\ z_0 + \xi + \varepsilon_j\}$ while the information set of the long term investors becomes $\Omega_1 = \{P_0, P_1,\ \theta_1 + d + v_i,\ z_1 + \xi + \zeta_i\}$. In other words, the private of signals of the short term traders and investors are correlated. In particular we are now allowing the investors, just like the short-term traders, to observe a noisy sunspot signal correlated with the sunspot signals received by short-term traders. Here d and ξ are common information both for short term traders and long term investors. θ_0 and z_0 however are in the private information sets of the short-term traders, and the investors can only learn about them from observing the market price. We assume z_0, ξ and z_1 are drawn from standard normal distributions and ζ_i are drawn from a normal distribution with mean of 0 and variance of σ_ζ^2. The equilibrium conditions are again given by (21) and (22). We now can show that there exists an continuum of sunspot equilibria indexed by $0 \le \sigma_z^2 \le \frac{1}{4}\sigma_{\theta_0}^2$, with the prices

$$\log P_0 = \phi\theta_0 + \sigma_z z_0 + d, \quad (28)$$
$$\log P_1 = \phi\theta_0 + \sigma_z z_0 + d + \theta_1, \quad (29)$$

where ϕ is given by

$$0 \leq \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_z^2}{\sigma_{\theta_0}^2}} \leq 1. \tag{30}$$

The proof is very similar to the that of Proposition 4 and hence is omitted.[9] A conclusion we can draw is that the market is in general not fully efficient in aggregating the information of the short-term traders, even if the investors receive sunspot and dividend signals correlated with the private signals of short-term traders. As long as the short-term traders as a whole have some private information, there exists sunspot equilibria.

5 Alternative Market Structures

In our benchmark model, only short term traders are present in period 0 market. We first relax that assumption. Suppose now that both short term traders and long term investors are present in period 0, but only long term investors are present in period 1. The short term traders maximize their expected payoff in period 1 and the long term investors maximize their expected payoff in period 2. Let Ω_0 denote the information set of a particular short term trader j. The short term trader's utility maximization problem is given by

$$\max_{x_{j0} \in (-\infty, +\infty)} \mathbb{E}[(P_1 - P_0)\, x_{j0} + P_0 + w | \Omega_0].$$

and the first order condition is

$$P_0 = \mathbb{E}[P_1 | \Omega_0] \tag{31}$$

The long term investors trade in both periods. Let x_{i0}, and x_{i1} be the asset holding of the long term trader in period 0 and 1 respectively and let B_{i0} and B_{i1} be the bond holding of such investors in period 0 and 1 respectively. The long term investors try to maximize their consumption in period 2. Then the budget constraints for investor i are

$$\begin{aligned} P_0 x_{i0} + B_{i0} &= w \\ P_1 x_{i1} + B_{i1} &= P_1 x_{i0} + B_{i0}, \\ C_{i2} &= D x_{i1} + B_{i1}, \end{aligned}$$

The long term investor's problem can be solved recursively. Let Ω_0^* and Ω_1 denote the information set of a particular investor i in period 0 and 1, respectively. Note that $\Omega_0^* \subseteq \Omega_1$. Given x_{i0} and B_{i0}, the utility maximization problem of the long term investor in period 1 then becomes:

$$\max_{x_{i1} \in (-\infty, +\infty)} \mathbb{E}\left\{ [P_1 x_{i0} + B_{i0} + (D - P_1) x_{i1}] \,| \Omega_1 \right\}.$$

[9] As d is commonly observed by both the investors and traders, its distribution does not matter. We can assume for example that $d \sim \mathcal{N}(-\frac{1}{2}\sigma_d^2, \sigma_d^2)$.

Applying the law of iterated expectations and substituting out B_{i0} by the budget constraint, we can write the period 0's problem as

$$\max_{x_{i0}\in(-\infty,+\infty)} \mathbb{E}\left\{\left[\left(P_1 - P_0\right)x_{i0} + w\right]|\Omega_0^*\right\}.$$

The first order conditions for investor i are

$$P_1 = \mathbb{E}[D|\Omega_1], \tag{32}$$

$$P_0 = \mathbb{E}[P_1|\Omega_0^*], \tag{33}$$

The asset market clearing conditions require $\int x_{j0}dj + \int x_{i0}di = 1$ and $\int x_{i1}di = 1$. We discuss several cases below.

Case 1: In the first case, we assume that only the short term trader has private information regarding D and the sunspots. Namely $\Omega_0 = [P_0, \theta + e_j, z + \varepsilon_j]$, $\Omega_0^* = P_0$ and $\Omega_1 = [P_0, P_1]$. In this case, Proposition 3 applies. It is easy to verify that $\log P_1 = \log P_0 = \phi\theta + \sigma_z z$ satisfies equations (31), (32) and (33).

Case 2: In this case, both the short term traders and the investors receive private information on the dividends and the sunspot in period 0. As in section 4.2, we assume that $\log D = \theta_0 + \theta_1 + d$ and that $z = z_0 + \xi + z_1$. We also assume that θ_0 and z_0 are private information to the short term traders, while d and ξ are common information for both short term traders and investors. In other words, the information for the short term trader in period 0 is $\Omega_0 = [P_0, \theta_0 + d + e_j, z_0 + \xi + \varepsilon_j]$, while the long term investor's information sets in period 0 and 1 are $\Omega_0^* = \{P_0, d + v_i, \xi + \zeta_i\}$ and $\Omega_1 = \{P_1,\ \theta_1,\ z_1\} \cup \Omega_0^*$.[10] Again it is easy to show that there exists an continuum of sunspot equilibria indexed by $0 \leq \sigma_z^2 \leq \frac{1}{4}\sigma_{\theta_0}^2$, with the prices

$$\log P_0 = \phi\theta_0 + \sigma_z z_0 + d, \tag{34}$$

$$\log P_1 = \phi\theta_0 + \sigma_z z_0 + d + \theta_1, \tag{35}$$

where ϕ is given by

$$0 \leq \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_z^2}{\sigma_{\theta_0}^2}} \leq 1. \tag{36}$$

Notice the price functions are exactly the same as in the section 4. In both cases, under the equilibrium prices both short term traders and long term investors are indifferent between holding stocks or bonds in period 0, and the long term investors are indifferent between holding stocks

[10] Other information partitions for the long term investors, for example: $\Omega_0^* = \{P_0, d, z_1 + \xi + \zeta_i\}$ and $\Omega_1 = \{P_1, \theta_1 + \nu_i\} \cup \Omega_0^*$ can support the same equilibrium. We can also allow noisy information in Ω_1, for example $\Omega_0^* = \{P_0, d + v_i, \xi + \zeta_i\}$ and $\Omega_1 = \{P_1,\ \theta_1 + v_i^*\ z_1 + \zeta_i^*\}$, to support the same equilibrium prices.

or bonds in period 1. We can then assume that $0 \le x_{j0} = x \le 1$, $x_{i0} = 1 - x$ and $x_{i1} = 1$ in a symmetric equilibrium. Our results are therefore robust to incorporating investors with private information in the early stages of trading.

6 Multiple Assets and Price Co-Movements

It is widely known that the traditional asset pricing models cannot explain why asset prices have a high covariance relative to the covariance of their fundamentals. (See Pindyck and Rotemberg (1993), Barberis, Shleifer and Wurgler (2005), and Veldkamp (2006).[11]) Excessive co-movement in the asset prices may be explained by correlated liquidity or noise trading across markets. However such excessive co-movement of asset prices is also present in markets dominated by institutional investors, less likely to behave as noise traders. For example, Sutton (2000) documents excessive co-movement in ten-year government bond yields between the United States, Japan, Germany, the United Kingdom and Canada that are not fully explained by the co-movments in economic activity, inflation or short term interest rates. These excessive co-movements in the bonds markets appear to be even stronger than in equity markets. Cappiello, Engle, and Sheppard (2006) find that the median bond-bond return correlation is 0.7276, and the median equity-equity correlation is 0.4435 for bond and equity indices across 21 countries. These observations suggest that co-movements of asset prices are present in markets where arbitrage by institutional investors may dominate liquidity or noise trading.

In this section, we show that asset prices driven by the sentiment or sunspot shocks can exhibit high co-movements even if their underlying fundamentals are uncorrelated without resorting to noisy trading. The model is similar to the benchmark model above, but with multiple assets. For simplicity we consider two assets, a and b. The two assets yield final dividends in period 2 given by:

$$\log D_{2\ell} = \theta_\ell, \text{for } \ell = a, b. \tag{37}$$

We assume that θ_ℓ, $\ell = a, b$ are drawn from same normal distribution with mean $-\frac{1}{2}\sigma_\theta^2$ and variance of σ_θ^2. To highlight the co-movement, we assume that $cov(\theta_a, \theta_b) = 0$. For simplicity we consider representative agents in each period. The trader in period 0 solves

$$\max_{x_{0a}, x_{0b}} \sum_{\ell=a,b} \{\mathbb{E}[P_{1\ell}|\theta_a, \theta_b, P_{0a}, P_{0b}] - P_{0\ell}\} \, x_{0\ell}, \tag{38}$$

where x_{0a}, x_{0b} are the asset holdings of the trader for asset a and b, respectively. Here $P_{1\ell}$ and $P_{0\ell}$

[11] Veldkamp (2006) constructs a model with markets for information to explain asset price co-movements which we discuss in Section 9.

are the asset $\ell's$ price in period 0 and 1. The investor in period 1 solves

$$\max_{\substack{x_{1a}\in(-\infty,+\infty),\\ x_{1b}\in(-\infty,+\infty)}} \sum_{\ell=a,b} \{\mathbb{E}[D_{2\ell}|P_{1a}, P_{1b}, P_{0a}, P_{0b}] - P_{1\ell}\} x_{1\ell}, \tag{39}$$

where $x_{1\ell}$ are the asset holding of investor in period 1 for asset a and b. The first order conditions are:

$$P_{0\ell} = \mathbb{E}[P_{1\ell}|\theta_a, \theta_b, P_{0a}, P_{0b}], \tag{40}$$

$$P_{1\ell} = \mathbb{E}[D_{2\ell}|P_{1a}, P_{1b}, P_{0a}, P_{0b}]. \tag{41}$$

Since the agents are risk neutral, they will be indifferent in buying the asset or not. We will focus on the symmetric equilibrium again, namely an equilibrium with $x_{0\ell} = 1, x_{1\ell} = 1$ for $\ell = a$ and b.

Proposition 5 *There exists a fully revealing equilibrium with* $\log P_{0\ell} = \log P_{1\ell} = \theta_\ell$.
Proof. *The proof is straightforward.* ■

Notice that in the fully revealing equilibrium the correlation of asset prices is zero and there is no co-movement of asset prices.

Proposition 6 *There exists a continuum of equilibria with prices fully synchronized among assets. The asset prices take the form*

$$\log P_{0\ell} = \log P_{1\ell} = \phi(\theta_a + \theta_b) + \sigma_z z + \frac{1}{2}\phi\sigma_\theta^2, \tag{42}$$

for $\ell = a$ *and* b. *Here we have* $0 \le \phi \le \frac{1}{2}$ *and*

$$\sigma_z^2 = \phi(1 - 2\phi)\sigma_\theta^2. \tag{43}$$

Proof. *Asset prices do not change in period 0 and 1. Hence equation (40) is satisfied automatically. We only need to insure equation (41) holds. Due to symmetry, it is sufficient to prove (41) holds for asset a. We need to show:*

$$\begin{aligned}
\phi(\theta_a + \theta_b) + \sigma_z z + \frac{1}{2}\phi\sigma_\theta^2 &= \log\exp\{\mathbb{E}[\theta_a|\phi(\theta_a + \theta_b) + \sigma_z z]\} \\
&= -\frac{1}{2}\sigma_\theta^2 + \frac{\phi\sigma_\theta^2}{2\phi^2\sigma_\theta^2 + \sigma_z^2}\left[\phi(\theta_a + \theta_b) + \sigma_z z + \phi\sigma_\theta^2\right] + \frac{1}{2}\left[\sigma_\theta^2 - \frac{\left(\phi\sigma_\theta^2\right)^2}{2\phi^2\sigma_\theta^2 + \sigma_z^2}\right] \\
&= \frac{\phi\sigma_\theta^2}{2\phi^2\sigma_\theta^2 + \sigma_z^2}\left[\phi(\theta_a + \theta_b) + \sigma_z z\right] + \frac{1}{2}\frac{\left(\phi\sigma_\theta^2\right)^2}{2\phi^2\sigma_\theta^2 + \sigma_z^2}.
\end{aligned} \tag{44}$$

Comparing terms we obtain $\phi\sigma_\theta^2 = 2\phi^2\sigma_\theta^2 + \sigma_z^2$*, or*

$$\sigma_z^2 = \phi(1-2\phi)\sigma_\theta^2. \tag{45}$$

Then we have $\frac{1}{2}\frac{(\phi\sigma_\theta^2)^2}{2\phi^2\sigma_\theta^2+\sigma_z^2} = \frac{1}{2}\phi\sigma_\theta^2$. ■

The intuition for asset price co-movements is straightforward. Since the same traders trade the two assets, the asset prices will be determined by the same information set. If prices are driven not only by fundamentals but also by sentiments, then the sentiment shocks of the traders will drive both asset prices. The sentiment-driven co-movements is also consistent with the finding of Sutton (2000) and Cappiello, Engle, and Sheppard (2006).

It is straightforward to extend the information structure to allow any degree of co-movement. For example, we can assume that the total dividend of asset ℓ is given by $\theta_\ell + d_\ell$ and the information sets are $\Omega_0 = \{\theta_a + d_a, \theta_b + d_b, P_{0a}, P_{0b}, z\}$ and $\Omega_1 = \{d_a, d_b, P_{0a}, P_{0b}, P_{1a}, P_{2b}\}$. Here d_ℓ is the common information of the dividend observed by both short term traders and long term investors. We assume that $cov(d_a, d_b) = 0$ and $\sigma_d^2 = cov(d_a, d_a)$. Hence the total dividend of these two assets are not correlated. Then we can construct equilibria with prices

$$\log P_{0\ell} = \log P_{1\ell} = \phi(\theta_a + \theta_b) + \sigma_z z + \frac{\phi\sigma_\theta^2}{2} + d_\ell. \tag{46}$$

where ϕ and σ_z are given by Proposition (6). If $\sigma_d^2 > 0$, then the asset prices do not co-move perfectly with each other. When $\sigma_d^2/\sigma_\theta^2$ approaches infinity, the correlation between the asset prices becomes zero. Therefore we can always set the value of σ_d^2 to fit the observed covariance of asset prices. Notice that if $\sigma_d^2/\sigma_\theta^2$ increases, then the covariance of asset prices declines. This could be the result of a reduction in the information acquisition cost facing uninformed long term investors. Legal reform on disclosure requirements for example can also produce more information for the uninformed investors. Fox, Durnew, Morck and Yeung (2003) show that the enactment of new disclosure requirements in December 1980 caused a decline in co-movements, consistent with our theory.

7 Multi-Period Assets

We now extend our model to multiple periods to show that our sentiment driven equilibrium persists in a dynamic setting.[12] Along the lines initially pointed out by Tirole (1982), we also want to show

[12] It is important to study whether sentiment driven equilibrium will emerge in a dynamic setting as it has been shown that in some dynamic settings (although not in general) learning from prices can generate a full information equilibrium in the limit (see chapter 7 of Vives (2008) for a literature review).

that the sentiment-driven asset prices in our model look like an efficient market in the following sense: if an econometrician studies the asset data generated by the sunspot equilibria, they will find that the asset prices movements will be a random walk and not reject the efficient market hypothesis.

Suppose an asset created in period 0 yields a return or dividend only in period $T+1$. Between period 0 and $T-1$, a continuum of short term traders can trade the asset each period. Short term traders in each period hold the asset only for making capital gains. As in section 3 where there are only three periods, given prices private signals that traders receive do not matter, so we focus on a representative trader in each period. The final dividend is given by

$$\log D_{T+1} = \left(\sum_{t=0}^{T} \theta_t \right)$$

Again we assume that θ_t is indepdently drawn from a normal distribution with mean of $-\frac{1}{2}\sigma_\theta^2$ and variance of σ_θ^2. So the unconditional mean of D_{T+1} is given by 1. Denote the information set of traders in period $t = 0, 1, ..T$ as Ω_t, and their asset holding from period t to $t+1$ as x_t. Their maximization problem can be written as

$$\max_{x_t \in (-\infty, +\infty)} \left[\mathbb{E}\left(P_{t+1} | \Omega_t \right) - P_t \right] x_t, \tag{47}$$

for $t = 0, 1, ..T-1$. Investors who purchase the asset in period T solve

$$\max_{x_T \in (-\infty, +\infty)} \left[\mathbb{E}\left(D_{T+1} | \Omega_T \right) - P_T \right] x_T. \tag{48}$$

We assume short term traders in period t know θ_t, which may be interpreted as trader t's the private information regarding the final dividend of the underlying asset. Since all the agents observe the past and current price, their information Ω_t is given by $\Omega_t = \{\theta_t, z_t\} \cup \{\cup_{\tau=0}^{t} P_\tau\}$, where z_t are i.i.d draws from the standard normal distribution representing sentiment shocks of traders born period t as in our benchmark model.

An equilibrium is a set of price functions $\{P_t\}_{t=0}^{T}$ such that $x_t = 1$ solves (47) for $\tau = 0, 1, ..T-1$ and $x_T = 1$ solves (48), where equilibrium conditions for individual optimization are given by

$$\mathbb{E}\left(P_{t+1} | \Omega_t \right) = P_t, \text{for } t = 0, 1, ..T-1 \tag{49}$$

and

$$\mathbb{E}\left(D_{T+1} | \Omega_T \right) = P_T. \tag{50}$$

Proposition 7 $P_t = \exp(\sum_{\tau=0}^{t} \theta_\tau)$ *for* $t = 0, 1, ...T$ *is always an equilibrium.*

Proof. *The proof is straightforward. The information of the past prices reveal the history of* $\{\theta_\tau\}_{\tau=0}^{t-1}$*, It is easy to check that*

$$\begin{aligned}
\exp(\sum_{\tau=0}^{t}\theta_\tau) &= \mathbb{E}_t\left[\exp(\sum_{\tau=0}^{t}\theta_\tau+\theta_{t+1})\right] \\
&= \exp(\sum_{\tau=0}^{t}\theta_\tau). \qquad (51)
\end{aligned}$$

So (49) is satisfied for $\tau = 0, 1, ...T-1$*, where we have utilized that fact* $\mathbb{E}_t \exp(\theta_{t+1}) = 1$*. Finally by construction, equation (50) is automatically satisfied.* ■

In the above equilibrium, the price will eventually converge to the true fundamental price. The market is dynamically efficient in the sense that all private information is revealed sequentially by the market prices. However as in the benchmark model, the above equilibrium is not the only one. Assume that traders at each t condition their expectations of the price P_t some sentiment or sunspot shock z_t that they receive. We assume that θ_t, z_t are only observed by the traders at t. We have the following Proposition regarding equilibrium price.

Proposition 8 *There exists a continuum of sentiment-driven or sunspot equilibria indexed by* $0 \leq \sigma_z \leq \frac{1}{4}\sigma_\theta^2$*, with the price in period* t *given by*

$$\log P_t = \sum_{\tau=0}^{t}(\phi\theta_\tau + \sigma_z z_\tau), \qquad (52)$$

for $t = 0, 1, 2, ...T-1$ *and*

$$\log P_T = \theta_T + \sum_{\tau=0}^{T-1}(\phi\theta_\tau + \sigma_z z_\tau). \qquad (53)$$

and

$$0 \leq \phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_z^2}{\sigma_\theta^2}} \leq 1. \qquad (54)$$

Proof. *We first prove that (53) holds. For the investor in period* $T-1$*, equation (49) requires*

$$\begin{aligned}
\sum_{\tau=0}^{T-1}(\phi\theta_\tau + \sigma_z z_\tau) &= \log \mathbb{E}\left[\exp\left(\log P_T\right)|\Omega_{T-1}\right] \\
&= \sum_{\tau=0}^{T-1}(\phi\theta_\tau + \sigma_z z_\tau) + \log \mathbb{E}\left[\exp(\theta_T)|\Omega_{T-1}\right]. \qquad (55)
\end{aligned}$$

Notice $\mathbb{E}\left[\exp(\theta_T)|\Omega_{T-1}\right] = 1$. *So the above requirement is satisfied. We then prove that equation (49) holds for* $t = 0, 1, 2, ..T-2$. *Given the price structure, the information set* Ω_t *is now equivalent to* $\tilde{\Omega}_t = \{z_t\} \cup \{\theta_t\} \cup \{\phi\theta_\tau + \sigma_z z_\tau\}_{\tau=0}^{t-1}$. *Equation (49) can then be re-written as*

$$\log P_t = \log \mathbb{E}[\exp\left(\log P_{t+1}\right)|\tilde{\Omega}_t]. \tag{56}$$

Plugging in the expression of $\log P_{t+1}$, *we obtain*

$$\begin{aligned}
\log P_t &= \mathbb{E}\left\{\left[\sum_{\tau=0}^{t}(\phi\theta_\tau + \sigma_z z_\tau) + \phi\theta_{t+1} + \sigma_z z_{t+1}\right]|\tilde{\Omega}_t\right\} + \frac{1}{2}\left(\phi^2\sigma_\theta^2 + \sigma_z^2\right) \\
&= \sum_{\tau=0}^{t}(\phi\theta_\tau + \sigma_z z_\tau) - \frac{\phi}{2}\sigma_\theta^2 + \frac{1}{2}\left(\phi^2\sigma_\theta^2 + \sigma_z^2\right) \\
&= \sum_{\tau=0}^{t}(\phi\theta_\tau + \sigma_z z_\tau).
\end{aligned} \tag{57}$$

where the third line comes from the fact $\phi^2\sigma_\theta^2 + \sigma_z^2 = \phi\sigma_\theta^2$ *by exploiting equation (54). So Equation (49) holds for* $t = 0, 1, 2, ..T-2$. *Finally for the investor of period* T, *we have*

$$P_T = \mathbb{E}\left[D_{T+1}|\Omega_T\right], \tag{58}$$

where Ω_T *is equivalent to* $\tilde{\Omega}_T = \{\theta_T,\, z_T\} \cup \{\phi\theta_\tau + \sigma_z z_\tau\}_{\tau=0}^{T-1}$, *learned from observing past prices. Notice* $\tilde{\Omega}_T$ *does not directly contain any past realization of* θ_t *or* z_t, *assumed to be private information of the trader period* $t \leq T-1$. *The above equation then yields*

$$\begin{aligned}
\log P_T &= \theta_T + \sum_{t=0}^{T-1}\left\{\mathbb{E}\left[\theta_t|\phi\theta_t + \sigma_z z_t\right] + \frac{1}{2}var(\theta_t|\phi\theta_t + \sigma_z z_t)\right\} \\
&= \theta_T + \sum_{t=0}^{T-1}\frac{\phi\sigma_\theta^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\left(\phi\theta_t + \sigma_z z_t + \frac{\phi}{2}\sigma_\theta^2\right) - \frac{T}{2}\sigma_\theta^2 + \frac{T}{2}\left[\sigma_\theta^2 - \frac{\left(\phi\sigma_\theta^2\right)^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\right] \qquad (59) \\
&= \theta_T + \sum_{t=0}^{T-1}\frac{\phi\sigma_\theta^2}{\phi^2\sigma_\theta^2 + \sigma_z^2}\left(\phi\theta_t + \sigma_z z_t\right),
\end{aligned}$$

we now simplify the above equation to obtain equation (53). ■

Prices are driven by the sentiment of the short term traders. Note that equation (52) implies that the asset price follows a random walk in the sentiment-driven equilibria. Although the efficient market hypothesis and the random walk of asset prices are not identical concepts, most tests of

EMH focus on the predictability of asset prices: if the market is efficient then rational investors will immediately react to informational advantages so that the profit opportunities are eliminated. As a result, information will be fully revealed by asset prices, and all subsequent price changes will only reflect new information. In other words, future asset prices are unpredictable. Therefore market efficiency and unpredictability are not equivalent. If an econometrician studies the asset price driven by the sentiment shocks in our model, he will conclude that asset prices are unpredictable. Yet, the sentiment shocks can generate permanent deviations of asset prices from their fundamental value.

8 Implementing the Equilibria

So far we relied on rational expectations to characterize equilibrium prices and showed that fully revealing REE exist. Vives (2014) however raises the issue that in fully revealing rational expectation equilibrium may not be implementable: if short term traders ignore their private signal at fully revealing equilibrium prices, we are faced with the problem of how fully revealing equilibrium prices incorporating these private signals are in fact realized in the market. Since Vives' critique applies in our model, we study implementability of our equilibria. For simplicity we focus on our baseline three-period model in section 3.1; the equilibria in other extended models can be implemented in the same way.

We first note that the sentiment drive equilibrium with $\log P_1 = \log P_0 = \phi\theta + \sigma_z z$ exist even if the signals are perfect, namely when $\sigma_e^2 = 0$ and $\sigma_\varepsilon^2 = 0$. To fix ideas, we first consider this simple special case. As the fully revealing equilibrium is a special case of the sentiment-driven equilibrium, we can focus on the implementability of the sentiment-driven equilibria. We follow Vives (2014) to implement the equilibria in our model by constructing demand functions for short term traders and the investors. To construct well-defined demand functions with risk neutral agents, we assume that the trading is limited by $0 \leq x_{j0} \leq \bar{x}$[13], where $\bar{x} > 1$ is the maximum asset holding for all agents. The equilibrium can then be implemented by constructing demand functions as follows. Every the short-term trader believes that the price in next period is given by $\log P_1 = \phi\theta + \sigma_z z$, and submits its demand function to a market auctioneer

$$x_{j0} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_0 < \phi\theta + \sigma_z z \\ 1 & \text{if } \log P_0 = \phi\theta + \sigma_z z \\ 0 & \text{if } \log P_0 > \phi\theta + \sigma_z z \end{array} \right\}, \tag{60}$$

The only price that can clear the market is $\log P_0 = \phi\theta + \sigma_z z$. Note since the agents know θ and z perfectly in this case, the equilibrium price is implementable. After observing the price

[13] It is easy to see that our results below are not affected if we assume $x_{j0} \geq -x_L$, where $-x_L > -\infty$ is the minimum asset holding for all agents.

$\log P_0 = \phi\theta + \sigma_z z$, the long term investors submit their demand function to an market auctioneer according to

$$x_{i1} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_1 < \log P_0 = \phi\theta + \sigma_z z \\ 1 & \text{if } \log P_1 = \log P_0 = \phi\theta + \sigma_z z \\ 0 & \text{if } \log P_1 > \log P_0 = \phi\theta + \sigma_z z \end{array} \right\} \tag{61}$$

Again the only price will clear the market, consistent with (16) and (19) is $\log P_1 = \phi\theta + \sigma_z z$. Since the long term trader can observe the first period price, the long term trader's demand function is well defined. This establishes that equilibrium with $\log P_1 = \log P_0 = \phi\theta + \sigma_z z$ is an indeed implementable rational expectation equilibrium.

Now we turn to the case with $\sigma_e^2 > 0$. We assume that z is drawn from normal distribution with mean 0 and variance σ_z. Note that a fully revealing equilibrium with $\log P_0 = \log P_1 = 0$ that does not depend on sentiments is always implementable. In this fully revealing equilibrium as well as in the sentiment-driven equilibria the short term traders ignore their private signals at the equilibrium prices, so as noted earlier, which raises the question of how equilibrium prices that aggregate private signals is in fact realized, a problem considered by Vives (2014) as the implementability REE. Vives (2014) assumes that the valuation of assets by each trader has a common as well as a private component. Each trader receives a private signal that bundles the common and private valuations together. In that case, although the price fully reveals the common component, private signals are still useful for providing information on private valuations. As a result, the demand functions for the traders depend on private signals. The demand functions and information contained in the private signals can then be aggregated to obtain the equilibrium price.

We now show how this mechanism proposed by Vives (2014) can also implement the equilibria in our economy. Suppose that the asset also yields a random idiosyncratic utility u_j drawn from a normal distribution with mean 0 and variance of σ_u^2 for the short term trader in period 1. One can think for example that the asset is a firm generating profits in both periods, and that the profits depend on the ability of the owner. If the short term traders' abilities are idiosyncratic and the long term investor's ability is homogenous, then we can generate payoffs for short-term traders that depend on idiosyncratic components. The short term traders receive signals $s_j = \theta + u_j + e_j$, and they solve the following problem:

$$\max_{x_{j0}} \mathbb{E}[(P_1 - P_0 + u_j)\, x_{j0} | \theta + u_j + e_j, P_0, z], \tag{62}$$

The problem of the long term investors is the same as in section 3.1 and therefore their first order condition is given by (13). Let us define $p_0 = \log(P_0 - \bar{u})$, where $\bar{u}$ is to be determined by market clearing in (66) below. Suppose all short term investors believe $p_0 = \log P_1 = \phi\theta + \sigma_z z$. Under such a belief $\frac{p_0 - \sigma_z z}{\phi} = \theta$ and $u_j + e_j = s_j - \frac{p_0 - \sigma_z z}{\phi}$. Therefore their demand function depends on their estimate of u from the signal $s_j - \frac{p_0 - \sigma_z z}{\phi}$ relative to the threshold $\bar{u}$:

$$x_{j0} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2}(s_j - \frac{p_0-\sigma_z z}{\phi}) > \bar{u} \\ 1 & \text{if } \frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2}(s_j - \frac{p_0-\sigma_z z}{\phi}) = \bar{u} \\ 0 & \text{if } \frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2}(s_j - \frac{p_0-\sigma_z z}{\phi}) < \bar{u} \end{array} \right\} \tag{63}$$

Denote $\Phi(\cdot)$ as the cumulative distribution function of the standard normal. The aggregate demand function is therefore given by

$$x_0(p_0) = \bar{x} \int_{s_j > \bar{u}\frac{\sigma_u^2+\sigma_e^2}{\sigma_u^2} + \frac{p_0-\sigma_z z}{\phi}} dj = \bar{x} \left[1 - \Phi \left(\frac{\bar{u}\frac{\sigma_u^2+\sigma_e^2}{\sigma_u^2} + \frac{p_0-\sigma_z z}{\phi} - \theta}{\sqrt{\sigma_u^2+\sigma_e^2}} \right) \right], \tag{64}$$

where the second equality has used the fact that $s_j \sim N(\theta, \sigma_u^2 + \sigma_e^2)$ for given realization of θ. We can solve for p_0 and $\bar{u}$ using the market clearing condition, namely $x_0(p_0) = 1$. Then the price function

$$p_0 = \phi\theta + \sigma_z z, \tag{65}$$

and

$$1 - \Phi(\bar{u}\frac{\sqrt{\sigma_u^2+\sigma_e^2}}{\sigma_u^2}) = \frac{1}{\bar{x}}, \tag{66}$$

clear the market for any realization of the θ and z.

Finally the demand function for the long term trader is

$$x_{i1} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_1 < p_0 = \phi\theta + \sigma_z z \\ 1 & \text{if } \log P_1 = p_0 = \phi\theta + \sigma_z z \\ 0 & \text{if } \log P_1 > p_0 = \phi\theta + \sigma_z z \end{array} \right\} \tag{67}$$

Note that in the second period the market will clear if $\log P_1 = p_0 = \phi\theta + \sigma_z z$ and the first order condition (13) will be satisfied. Thus as in Proposition 3 we have a continuum sentiment-driven rational expectations equilibria that are implementable. However if σ_u^2 becomes zero the equilibrium is difficult to implement. In that case $\bar{u} = 0$ and $\frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2} = 0$, so the demand for each trader is simply $x_{j0} = 1$ regardless of their private signals. In this case, it is difficult to see how the equilibrium prices would be implemented or would incorporate the information in the private signals.

Alternatively, to implement the sentiment driven equilibria with a noisy signal, we can also follow the approach of Golosov, Lorenzoni and Tsyvinski (2014) by assuming decentralized trading in period 0. In particular, we can divide the first period into N sub-periods and allow $N \to \infty$: the short term traders randomly meet and trade bilaterally. In so doing they learn information from other traders through their own trading history. Eventually at the end of first period, all short term traders become perfectly informed about θ. We have already shown that the equilibrium can be implemented in this case. Finally following Vives (2008)[14] and Golosov, Lorenzoni and Tsyvinski (2014), we construct an approach based on learning to implement the fully revealing equilibrium in our model in the Appendix. In this extended model, the first period is divided into N sub-periods, with $N \to \infty$. The short term trader with index $j \in [\frac{\tau-1}{N}, \frac{\tau}{N}]$ is assumed to trade only twice, first in τth sub-period of period 0 and then in period 1. We assume that short term traders with index $j \in [\frac{\tau-1}{N}, \frac{\tau}{N}]$ receive the same private signal about θ, $s_{j0} = \theta + e_{\tau,0}$. In each sub-period τ of the initial period 0, the signal s_{j0} and past history of prices map into the price in that sub-period, P_{j0}. As the equilibrium price in each sub-period reflects the common information of the short term traders, it will be implementable. So the sequence of prices in period 0 will eventually reveal θ. In the appendix we show that the sentiment driven equilibrium can be also implemented similarly.

9 Some Related Literature

To put our results in context we now briefly discuss some papers based on informational asymmetries or restricted participation that are related to ours.

The sunspot equilibria that we have considered are not randomizations over multiple fundamental equilibria. Instead they are related to the early sunspot results of Cass and Shell (1983).[15] As in our model, Cass and Shell (1983) have a finite overlapping generations economy with a unique fundamental equilibrium. There are two periods, uniform endowments and the agents have separable utility functions defined over the two commodities. The consumers in the initial period are born before the commonly observed sunspot activity is revealed, and can trade with each other on the market for securities with payoffs contingent on the outcome of the extrinsic random variable determined by sunspot activity. There are also consumers born in the second period, after the sunspot is realized. Both generations of traders can then trade commodities on the spot market. In addition to the unique certainty equilibrium, Cass and Shell (1983) show the existence of a sunspot equilibrium with the relative commodity prices driven by extrinsic uncertainty. This rational expectations equilibrium arises from state contingent trades based on sunspot probabilities that create wealth effects. This mechanism differs from ours where a continuum of sunspot equilibria arise under private signals from the signal extraction problem as agents optimally disentangle the price signal into the fundamental and sunspot components.[16]

[14] Chapter 7 of Vives (2008) presents an excellent summary on learning and convergence to a full information equilibrium.

[15] See their appendix.

[16] Peck and Shell (1991) also show the existence of sunspot equilibria in a finite economy with a unique fundamental

Allen, Morris and Shin (2006) also study a model structure similar to ours, with an overlapping generations of traders who each live for two periods. A new generation of traders of unit measure is born at each date t. When the traders are young, they receive a noisy signal about the liquidation value of the asset at terminal time T, and they trade the asset to build up a position in the asset, but do not consume. In the next period when they are old, they unwind their asset position to acquire the consumption good, consume, and die. The asset supply each period is stochastic and unobserved, which prevents the equilibrium prices to reveal the terminal liquidation value. Allen, Morris and Shin (2006) show that the law of iterated expectations for average expectations each period can fail, so that market prices may be systematically lower than average expectations. This only happens if the traders are risk averse and their signals are imprecise. With risk-neutral short-term traders, prices again become fully revealing. Unlike our model where short-term traders condition their portfolio decisions on both fundamental and sunspot signals that gives rise to correlated actions, in Allen, Morris and Shin (2006) agents condition their trades on fundamentals alone, so the issue of multiple sunspot equilibrium does not arise.

Cespa and Vives (2014) however show that the presence of liquidity or noise traders with correlated demands across periods will tend to decrease price volatility, and induce risk-averse short-term traders to trade more aggressively based on fundamentals, as discussed in the Introduction. Cespa and Vives (2014) find that, in their high information equilibrium, this effect can moderate the bias in prices arising from the failure of the law of iterated expectations noted by Allen, Morris and Shinn (2006), and can bring equilibrium prices closer to fundamentals.

Our results are closely related to the Angeletos, Lorenzoni and Pavan (2010), who explore a related market structure where entrepreneurs are receive noisy private signals about the ultimate return to their investments. Entrepreneurs are also aware that they are collectively subject to correlated sentiments of market optimism or pessimism about investment returns. These sentiments are embodied in a second correlated noisy signal that introduces non-fundamental noise into entrepreneurs' investment decisions. The traders buy the assets from entrepreneurs without observing their signals, but they do observe the aggregate level of investment. This observation induces a signal extraction problem for the traders as aggregate investment is now driven by the fundamentals of investment returns as well as non-fundamentals. Angeletos, Lorenzoni and Pavan (2010) show that the resulting correlated market sentiments can introduce amplification of the noise on fundamentals as well as self-fulfilling multiple equilibria. In such equilibria traders are willing to purchase assets at "speculative" prices consistent with price expectations of entrepreneurs that differ from

equilibrium by allowing non-Walrasian trades prior to trading on the post- sunspot spot markets. Spear (1989) shows the possibility of sunspots with a unique fundamental equilibrium for an OLG model with two islands where prices in one island act as sunspots for the other and vice versa. Finally, in OLG models the non-monetary equilibrium may be considered unique fundamental equilibrium while monetary equilibria may be viewed as non-fundamental bubbles as in Tirole (1982).

expectations on fundamentals. The correlation in entrepreneurial investment decisions is similar to the correlated decisions of the short and long term traders in our model. They are induced by the sunspot driven prices and give rise to sunspot equilibria distinct from the price-revealing Grossman-Stiglitz equilibrium. An essential component of the multiplicity therefore stems from the correlated actions induced by non-fundamentals, yielding additional "correlated equilibria" as discussed earlier by Aumann (1987), Aumann Peck and Shell (1988) and Maskin and Tirole (1987).[17]

Other recent papers in the literature have also explored the role of informational asymmetries and costly information to generate price movements that diverge from fundamentals in order to explain market data, without incorporating non-fundamentals into the information structures of markets that generate multiple rational expectations equilibria and sunspot fluctuations.

For example, in Albagli, Hellwig and Tsyvinski (2011), prices diverge from expected dividends from the perspective of an outside observer. Their model has noise traders as well as risk neutral informed and uninformed traders facing limits on their asset positions. Under risk neutrality and heterogenous beliefs driven by private signals, market clearing prices are determined by the marginal trader whose noisy private signal makes her indifferent between trading or not. Thus fluctuations in demand coming from realizations of fundamentals, or from noise traders, alters the identity of the marginal investor. This drives a wedge between prices and expected returns from the perspective of an outsider, and generates excess price volatility relative to fundamentals. The equilibrium is nevertheless unique since, unlike our model, price expectations and investment decisions are not conditioned on non-fundamentals or sunspots.

Our results on the co-movement of asset prices in excess of the fundamentals are also related to those of Veldkamp (2006) who introduces multiple assets with correlated payoffs and information markets into the model of Grossman and Stiglitz (1980). The stochastic supplies of risky assets are unobserved and prevents prices from being fully revealing, as would also be the case in the presence of noise traders. Introducing information markets creates strategic complementarities in information acquisition. Since information is produced with both fixed and variable costs, as more agents purchase the same information, the average cost of information is reduced. In such an economy with increasing returns to information production, information producers will supply and investors will purchase the signals that yield information on multiple assets. The co-movement is produced as investors use a common subset of signals to predict the price of different assets. Unlike our model, in Veldkamp (2006) there are no short term traders and the information obtained by traders pertains only to fundamentals, so additional equilibria that can emerge when trader's price expectations can also depend on non-fundamental sunspots do not arise.

[17] Maskin and Tirole (1987) study a simple finite two period endowment economy with a unique equilibrium that can yield additional sunspot equilibria under correlated private signals, provided one of the goods is inferior. Such inferiority is not present in the models we consider. Benhabib, Wang and Wen (2012) also use the idea of correlated signals arising via sunspots in the context of s to induce a continuum of equilibria in a Keynesian macroeconomic model.

10 Conclusion

We study a market where sequential short term traders have private information and earn capital gains by trading a risky asset before it yields dividends, while uninformed investors purchase the asset for its dividend yield, forming expectations based on observed prices. In a rational expectation equilibrium, prices based on fundamentals can reveal the information of private traders. However we show that there are also rational expectations equilibria driven by sunspots. We show that our results on sunspot equilibria are robust to a wide range of on informational assumptions and market structures. If an econometrician studies the asset data generated by these sunspot equilibria, they will find that the asset prices follow a random walk that look as if they are generated by an efficient market reflecting fundamental values.

Our sentiment-driven asset prices under informational frictions are closely related to the recent literature that examine sentiment-driven business cycles (see, for example, Angeletos and La'O (2012) and Benhabib, Wang and Wen (2012)). The recent financial crisis suggests that asset price movements can have considerable impact on macroeconomic fluctuations. Future work may more closely explore the connections between sentiment-driven asset prices and macroeconomic fluctuations.

A Appendices

In this appendix, we show in more detail how the fully revealing equilibrium can be implemented through a sequence of trades in the first period, as in Vives (2008) and Golosov, Lorenzoni and Tsyvinski (2014). The first period is divided into N sub-periods[18]. The short term traders trade the asset the sequentially. Let $\tau = 1, 2, ..., N$ be the τth sub-period in period 0. If we think of a period is a month or year, then the sub-periods may be trading days, hours or minutes. The short term trader with index $j \in [\frac{\tau-1}{N}, \frac{\tau}{N}]$ is assumed to trade only twice, first in $\tau' th$ sub-period of period 0, and then in period 1. We assume that each short-term trader with index $i \in [\frac{\tau-1}{N}, \frac{\tau}{N}]$ receives the same private signal about θ:

$$s_{j0} = [\theta + e_{\tau,0}, z + \varepsilon_{\tau,0}] \tag{A.1}$$

Finally we assume that the short term trader with index i can observe prices $(P_{1,0}, P_{2,0}, ...P_{i,0})$. The basic idea is that trading in these sub-periods will gradually incorporate the private information into prices. As the short term traders in the same sub-period have common information, equilibrium prices are implementable if they are functions of this common information.

Each short term trader solves

$$\max_{x_{j0}} E[(P_1 - P_{\tau,0})x_{j0}|P_{1,0}, P_{2,0}, ...P_{\tau,0}, \theta + e_{\tau,0}, z + e_{\tau,0}], \text{ for } j \in [\frac{\tau-1}{N}, \frac{\tau}{N}] \tag{A.2}$$

The first order condition is then given by

$$P_{\tau,0} = E[P_1|P_{1,0}, P_{2,0}, ...P_{\tau,0}, \theta + e_{\tau,0}, z + \varepsilon_{\tau,0}] \tag{A.3}$$

The long-term investor solves

$$\max_{x_{i1}} E[(\exp(\theta) - P_1)x_{i,1}|P_{1,0}, P_{2,0}, ...P_{N,0}, P_1], \tag{A.4}$$

which yields the interior first order condition:

$$E[\exp(\theta)|P_{1,0}, P_{2,0}, ...P_{N,0}, P_1] = P_1. \tag{A.5}$$

The market clears in each sub-periods and period 2, so we also have

$$\int_{j\in[\frac{\tau-1}{N}, \frac{\tau}{N}]} x_{j0} dj = \frac{1}{N} \text{ and } \int x_{i1} di = 1 \tag{A.6}$$

[18] We can also divide the second period into N sub-periods. The results will be the same since long-term traders have the same information. It does not matter whether they trade sequentially or not.

Proposition 9 *$P_{\tau,0} = P_1 = 1 = E\exp(\theta)$ for $\tau = 1, 2, .., N$ is always an equilibrium.*

Proof. *The proof is straightforward.* ■

The equilibrium is implementable since all agents knows the distribution of θ.

We now consider the fully revealing equilibrium. For simplicity, we will only consider the limiting case $N \to \infty$. We have the following Proposition.

Proposition 10 *There is another equilibrium with $\log P_1 = \theta$ and*

$$\log P_{\tau,0} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{0,\ell}), \tag{A.7}$$

where $\bar{p}_{0\tau}$ is a constant. In the other words, prices become fully revealing asymptotically.

Proof. *First we notice that*

$$\begin{aligned} E\left[\theta|\theta + e_{1,0}, \theta + e_{2,0}, ..., \theta + e_{\tau,0}\right] &= -\frac{1}{2}\sigma_\theta^2 + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{1}{\tau}\sigma_e^2}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} - \frac{1}{2}\sigma_\theta^2), \\ var(\theta|\theta + e_{1,0}, \theta + e_{2,0}, ..., \theta + e_{\tau,0}) &= \sigma_\theta^2 - \frac{\tau\sigma_\theta^2}{\tau\sigma_\theta^2 + \sigma_e^2}\sigma_\theta^2 = \sigma_\theta^2 - \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{1}{\tau}\sigma_e^2}\sigma_\theta^2. \end{aligned}$$

So

$$\log E[\exp(\theta)|\theta + e_{1,0}, \theta + e_{2,0}, ..., \theta + e_{\tau,0}] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{1}{\tau}\sigma_e^2}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}).$$

Second, under the belief $\log P_1 = \theta$, we have

$$\log P_{1,0} = \log E[\exp(\theta)|\theta + e_{1,0}] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_e^2}\left(\theta + e_{1,0}\right).$$

By observing $P_{1,0}$ and $\theta + e_{2,0}$, the short term trader in the second sub-period of period 0 effectively observes $\theta + e_{1,0}$ and $\theta + e_{2,0}$, so we have

$$\log P_{2,0} = \log E[\exp(\theta)|\theta + e_1, \theta + e_2] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_e^2}{2}}(\theta + \frac{e_{1,0} + e_{2,0}}{2}).$$

By induction, we have for $\tau = 1, 2, ..., N$

$$\log P_{\tau,0} = \log E[\exp(\theta)|\theta + e_{1,0}, \theta + e_{2,0}, ..., \theta + e_{\tau,0}] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{1}{\tau}\sigma_e^2}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0})$$

Finally when $N \to \infty$ we then have $\lim_{N\to\infty} \log P_{N,0} = \theta$ with probability 1. So there is no uncertainty for the long-term trader in period 1. They will be willing to pay θ in period 1. ■

The equilibrium above is also implementable by constructing demand functions

$$x_{j0} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_{\tau,0} < \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}) \\ 1 & \text{if } \log P_{\tau,0} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}) \\ 0 & \text{if } \log P_{\tau,0} > \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}) \end{array} \right\}, \text{ for } j \in [\frac{\tau-1}{N}, \frac{\tau}{N}] \tag{A.8}$$

for the short-term trader in sub-period τ where $x_{i1} = \bar{x}$ if $\log P_1 < \theta$, $x_{i1} = 1$ if $\log P_1 = \theta$ and $x_{i1} = 0$ if $\log P_1 > \theta$.

Finally we look at the implementability of the sentiment driven equilibria. We will focus on the equilibrium with $\log P_1 = \phi\theta + z$. Notice that since z has variance σ_z^2, we can rewrite it as $\log P_1 = \phi\theta + \sigma_z \hat{z}$ as in Proposition 3, relabeling $\hat{z}$ as a random sentiment shocks drawn from the standard normal distribution.

Proposition 11 *If $\sigma_\varepsilon^2 < \frac{1}{4}\sigma_e^2$, there then exists a sentiment driven equilibrium with $\log P_1 = \phi\theta + z$, where z is the sentiment shock whose variance is endogenously given by $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$, and where $\phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_\varepsilon^2}{\sigma_e^2}}$. In addition,*

$$\log P_{\tau,0} = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}} \left[\phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{\sigma_z}{\tau}\sum_{\ell=1}^{\tau} \varepsilon_{\ell,0} \right]. \tag{A.9}$$

Proof. *We first prove that under the assumption $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$, the effective signals for the short term traders are $\phi(\theta + e_{\tau,0}) + (z + \varepsilon_{\tau,0})$ if they believe $\log P_1 = \phi\theta + z$. To see that, we note that $[\theta + e_{0,\tau}, z + \varepsilon_{0,\tau}]$ are equivalent to two orthogonal signals $[\phi(\theta + e_{\tau,0}) + (z + \varepsilon_{\tau,0}), \frac{\sigma_\varepsilon^2}{\phi\sigma_e^2}(\theta + e_{\tau,0}) - (z + \varepsilon_{\tau,0})] \equiv [s_\tau^1, s_\tau^2]$. We show, under the assumption $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$,*

$$cov(s_\tau^1, s_\tau^2) = \frac{\sigma_\varepsilon^2}{\sigma_e^2}(\sigma_\theta^2 + \sigma_e^2) - (\sigma_z^2 + \sigma_\varepsilon^2) = 0,$$

$$cov(s_\tau^2, \phi\theta + z) = \phi\frac{\sigma_\varepsilon^2}{\phi\sigma_e^2}\sigma_\theta^2 - \sigma_z^2 = 0,$$

and for any $\ell \neq \tau$

$$cov(s_\tau^2, s_\ell^1) = \phi\frac{\sigma_\varepsilon^2}{\phi\sigma_e^2}\sigma_\theta^2 - \sigma_z^2 = 0.$$

It is then easy to show that

$$E[\phi\theta + z|\phi(\theta + e_{1,0}) + z + \varepsilon_{1,0}, ..., \phi(\theta + e_{\tau,0}) + z + \varepsilon_{\tau,0}]$$

$$= -\frac{1}{2}\phi\sigma_\theta^2 + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}} \left[\phi(\theta + \frac{1}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0}) + (z + \frac{1}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0}) + \frac{1}{2}\phi\sigma_\theta^2\right],$$

and

$$Var[\phi\theta + z|\phi(\theta + e_{1,0}) + z + \varepsilon_{1,0}, ..., \phi(\theta + e_{\tau,0}) + z + \varepsilon_{\tau,0}]$$

$$= \phi^2\sigma_\theta^2 + \sigma_z^2 - \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}\left(\phi^2\sigma_\theta^2 + \sigma_z^2\right).$$

It follows that

$$\log E[\exp(\phi\theta + \sigma_z z)|\phi(\theta + e_{1,0}) + \sigma_z(z + \varepsilon_{1,0}), ..., \phi(\theta + e_{\tau,0}) + \sigma_z(z + \varepsilon_{\tau,0})]$$

$$= \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}\left[\phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{1}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0}\right].$$

Finally we note

$$\log P_{1,0} = \log E[\exp(\phi\theta + z)|\phi(\theta + e_{1,0}) + (z + \varepsilon_{1,0})] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}\left[\phi\theta + z + \phi e_{1,0} + \varepsilon_{1,0}\right].$$

So by observing $P_{1,0}$ and $\phi(\theta + e_{2,0}) + (z + \varepsilon_{2,0})$, the short-term trader effectively observes $\{\phi(\theta + e_{\ell,0}) + (z + \varepsilon_{\ell,0})\}_{\ell=1,2}$. By induction the effective information set for short term trader in sub-period τ is $\{\phi(\theta + e_{\ell,0}) + (z + \varepsilon_{\ell,0})\}_{\ell=1,2,..,\tau}$. We obtain (A.9) from (A.3). Finally when $\tau \to N \to \infty$, the price becomes $\lim_{\tau\to\infty}\log P_{\tau,0} = \phi\theta + z$. The long term traders can therefore infer $\phi\theta + \sigma z$ from the history of prices. The long term traders can also infer the combination of noisy shocks $\{\phi e_{\ell,0} + \varepsilon_{\ell,0}\}_{\ell=1,2,...}$ from the prices, which, however, are useless information for them. They still need to solve signal extract problem defined by (A.5). $E[\exp(\theta)|\phi\theta + z] = \phi\theta + z$ holds if $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$ and $\phi = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\sigma_\varepsilon^2}{\sigma_e^2}}$. ■

Notice, unlike the results in section 4.1, we now put a further restriction that pins down the variance of sentiment shocks σ_z^2 at the sentiment-driven equilibrium. As the long-term traders can observe more than one past prices, they can potentially infer the fundamental shocks and sentiment shocks from these prices. However if the precision of the two signals are the same, namely $\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_e^2} =$

$\frac{\sigma_z^2}{\sigma_z^2+\sigma_e^2}$ or $\sigma_z^2 = \frac{\sigma_\varepsilon^2}{\sigma_e^2}\sigma_\theta^2$, the mapping between prices and the signals becomes non-invertible. As a result the long term trader will still not be able to distinguish the fundamental shocks and sentiments, even if they observe an infinite history of past prices. Since in each sub-period, the short term traders knows $\phi\theta + \sigma_z z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{1}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0}$ and the long term trader knows $\phi\theta + \sigma_z z$. The equilibrium can implemented straightforwardly by the demand functions

$$x_{j0} = \left\{ \begin{array}{ll} \bar{x} & \text{if } \log P_{\tau,0} < \frac{\sigma_\theta^2}{\sigma_\theta^2 + \frac{\sigma_\varepsilon^2}{\tau}}(\phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{\sigma_z}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0}) \\ 1 & \quad\text{if } \log P_{\tau,0} = \phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{\sigma_z}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0} \\ 0 & \quad\text{if } \log P_{\tau,0} > \phi\theta + z + \frac{\phi}{\tau}\sum_{\ell=1}^{\tau} e_{\ell,0} + \frac{\sigma_z}{\tau}\sum_{\ell=1}^{\tau}\varepsilon_{\ell,0} \end{array} \right\}, \text{ for } j \in [\frac{\tau-1}{N}, \frac{\tau}{N}] \tag{A.10}$$

for the short term traders, and $x_{i1} = \bar{x}$ if $\log P_1 < \phi\theta + z$, $x_{i1} = 1$ if $\log P_1 = \phi\theta + z$ and $x_{i1} = 0$ if $\log P_1 > \phi\theta + z$.

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