# Notes on Asset Prices in the exchange/production economy

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# Asset Prices in the production economy

- Real business cycle models are arguably successful at mimicking the cyclical behavior of macroeconomic quantities.
- However,Mehra and Prescott (1985) show that utility specifications common in RBC models have counterfactual implications for asset prices.
- We invistigate the performance of the RBC model in mimicking the difference between the average return to stocks and bonds, or the risk premium.
- We starts with an endowment economy and then we move to the production economy.

# The Lucas Tree model

- An continuum of identical households with time-seperable utility function
$$U = E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \tag{1}$$
- Two assets. One asset is bond, the other asset is a tree.
- The tree yields a fruit sequence $D_t$ according to some distribution to be specified.
- The tree can be traded among households. Its price is $P_t$. The bonds return is $R_{ft}$ between period t to period t+1.

# The Household Problem

- The household maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \tag{2}$$

- with a consequence of constraint

$$C_t + Q_t s_{t+1} + \frac{B_{t+1}}{R_{ft}} = (Q_t + D_t) s_t + B_t \tag{3}$$

- where $B_t$ is the bond holding in the begining of period and $s_t$ is the stock holding.

# The Bellman Equation

- Notice the household has two state variables, $s_t, B_t$
- three variables need to be solved, $C_t, s_{t+1}, B_{t+1}$
- Also the price $Q_t, R_{ft}, D_t$ are also exogenous states to the households. These are aggregate states. We denote them $\Phi_t$.
- Define $V(s_t, B_t; \Phi_t)$ as $V_t(s_t, B_t)$ as the value function, we have

# The Bellman Equation

- Define $V(s_t, B_t; \Phi_t)$ as $V_t(s_t, B_t)$ as the value function, we have

$$V_t(s_t, B_t) = \max_{s_{t+1}, B_{t+1}, C_t} \{ \frac{C_t^{1-\gamma}}{1-\gamma} + \beta E_t V_{t+1}(s_{t+1}, B_{t+1})$$
$$+\lambda_t[(Q_t + D_t)s_t + B_t - C_t - Q_t s_{t+1} - \frac{B_{t+1}}{R_{ft}}]\} \quad (4)$$

- FOC yields

$$C_t^{-\gamma} = \lambda_t \quad (5)$$

$$\lambda_t Q_t = \beta E_t \frac{\partial V_{t+1}(s_{t+1}, B_{t+1})}{\partial s_{t+1}} \quad (6)$$

$$\lambda_t \frac{1}{R_{ft}} = \beta E_t \frac{\partial V_{t+1}(s_{t+1}, B_{t+1})}{\partial B_{t+1}} \quad (7)$$

# The Bellman Equation

- The Envelop theorem yields

$$\frac{\partial V_t(s_t, B_t)}{\partial B_t} = \lambda_t(Q_t + D_t) \tag{8}$$

and

$$\frac{\partial V_t(s_t, B_t)}{\partial s_t} = \lambda_t \tag{9}$$

- so we have two Euler equations below

$$\lambda_t \frac{1}{R_{ft}} = \beta E_t \lambda_{t+1} \tag{10}$$

and

$$\lambda_t Q_t = \beta E_t \lambda_{t+1}(Q_{t+1} + D_{t+1}) \tag{11}$$

# Equilibrium

- The equilibrium is a sequence of price and quantity $\{Q_t, R_{ft}, C_t, s_{t+1}, B_{t+1}\}$ such that given $\{Q_t, R_{ft}, D_t\}$, such that $\{C_t, s_{t+1}, B_{t+1}\}$ solves the individual problem and all market clears.
- This implies

$$s_{t+1} = 1 \tag{12}$$

- and

$$B_{t+1} = 0 \tag{13}$$

$$C_t = D_t$$

# Equilibrium

- So we have the price of the tree is given by

$$D_t^{-\gamma} Q_t = \beta E_t D_{t+1}^{-\gamma} [Q_{t+1} + D_{t+1}] \tag{14}$$

- and the return is

$$D_t^{-\gamma} = \beta \frac{1}{R_{ft}} E_t D_{t+1}^{-\gamma} \tag{15}$$

- The above two equations define

$$Q_t = Q(D_t); R_{ft} = R_f(D_t) \tag{16}$$

the prices as a function of dividend.

# Equilibrium

- If $D_t$ follows a Markovian Process, with the transitional probaiblity

$$\Pr(D_{t+1} = d_j | D_t = d_i) = \pi_{ij} \tag{17}$$

- We then can caculate interest rate and the stock price by

$$d_i^{-\gamma} = \beta R_i \sum \pi_{ij} d_j^{-\gamma} \tag{18}$$

- and the stock price as

$$d_i^{-\gamma} Q_i = \beta \sum \pi_{ij} d_j^{-\gamma} [Q_j + d_j] \tag{19}$$

# Equilibrium

- For an example, consider $d_i = [1, 2]$ and the and $\pi_{ij} = .5$, $\gamma = 1$, we have

$$R_1 = \frac{1}{\beta \frac{1}{2}[1^{-1} + 2^{-1}]} \tag{20}$$

and

$$R_2 = \frac{2^{-1}}{\beta \frac{1}{2}[1^{-1} + 2^{-1}]} \tag{21}$$

- and

$$Q_1 = \frac{\beta}{2}[Q_1 + 1 + \frac{1}{2}(Q_2 + 2)] \tag{22}$$

$$Q_2 \frac{1}{2} = \frac{\beta}{2}[Q_1 + 1 + \frac{1}{2}(Q_2 + 2)] \tag{23}$$

Which can be used to solve $Q_1$, $Q_2$.

# Equilibrium

- Generally the stock price equals to

$$Q_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \tag{24}$$

- In deriving the above result, we have used the transversality condition such that

$$E_t \lim_{j \to 0} \beta^j \lambda_{t+j} Q_{t+j} = 0 \tag{25}$$

# Bubble

- consider a sequence of $P_t$ which satisify

$$P_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} P_{t+1} \tag{26}$$

- We can verify that

$$\tilde{Q}_t = Q_t + P_t \tag{27}$$

$$\tilde{Q}_t = \beta E_t [\tilde{Q}_{t+1} + \tilde{D}_{t+1}] \tag{28}$$

- However for any $P_t \neq 0$, the price $\tilde{Q}_t$ can not support the equilibrium. We call $Q_t$ as the fundamental price of the asset and $P_t$ as the bubble component.

# Bubble

- To understand the results, we consider a deterministic case. Namely $D_t = \bar{D}$. In such case we

$$R_{ft} = \frac{1}{\beta};$$
$$Q_t = \frac{\beta}{1-\beta} D \tag{29}$$

- And the utility level for the household in the equilibrium path is

$$U = \frac{1}{1-\beta} u(D) \tag{30}$$

- Now consider $P_t = \frac{1}{\beta^t} P_0$ and the price $\tilde{Q}_t = \frac{\beta}{1-\beta} D + \frac{1}{\beta^t} P_0$. Clearly we have

$$\frac{\beta}{1-\beta} D + \frac{1}{\beta^t} P_0 = \beta[\frac{\beta}{1-\beta} D + D + \frac{1}{\beta^{t+1}} P_0] \tag{31}$$

or

$$\tilde{Q}_t = \beta[\tilde{Q}_{t+1} + D_{t+1}] \tag{32}$$

# Bubble

- We have already seen the price $\tilde{Q}_t = \frac{\beta}{1-\beta} D + \frac{1}{\beta^t} P_0$ also satisfy the household's Euler equation:

$$\tilde{Q}_t = \beta[\tilde{Q}_{t+1} + D_{t+1}] \tag{33}$$

- However transversality condition is violated

$$\lim_{j \to 0} \beta^j \lambda_{t+j} Q_{t+j} = u'(D) P_0 \neq 0 \tag{34}$$

for $P_0 \neq 0$.

- We consider the case with $P_0 > 0$.

# Bubble

- We consider the case with $P_0 > 0$. We want to prove that $B_{t+1} = 0, s_{t+1} = 1$ is not an equilibrium.
- Consider the household decide $B_{t+1} = Q, s_{t+1} = 0$, and $C_0 = D + P_0$. In such case, household enjoy utility level

$$C_0 = D + P_0 \tag{35}$$

$$C_1 = [\frac{1}{\beta} - 1]Q = \frac{\beta}{1-\beta}D \times \frac{1-\beta}{\beta} = D \tag{36}$$

$$C_t = D \text{ for t} \geq 1 \tag{37}$$

# Bubble

- so we have

$$\tilde{U} > \frac{1}{1-\beta} u(D) \tag{38}$$

- so everyone would want to sell the stock in period 0. So for any $P_0 > 0$ can not be supported by the equilibrium.
- Generally speaking, the only price for the stock is

$$Q_t = E_t \sum \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \tag{39}$$

- So there can not be bubbles.

# Equity Premium

- Now we consider the equity premium in the model. To obtain a closed form solution, we assume that $logD_t$ follows a random walk. Namely

$$D_{t+1} = D_t \xi_t \tag{40}$$

- In this case,

$$Q_t D_t^{-\gamma} = \beta E_t D_{t+1}^{-\gamma} [Q_{t+1} + D_{t+1}] \tag{41}$$

- Guess a solution such that $Q_t / D_t = \kappa$, we can solve

$$\kappa D_t^{1-\gamma} = \beta(\kappa + 1) E_t D_{t+1}^{1-\gamma} \tag{42}$$

- or we have

$$\frac{\kappa + 1}{\kappa} = \frac{1}{\beta E_t \xi_{t+1}^{1-\gamma}}$$

# Equity Premium

- The risk free rate is

$$D_t^{-\gamma} = \beta R_{ft} E_t D_{t+1}^{-\gamma} \tag{43}$$

- so we have

$$R_{ft} = \frac{1}{\beta E_t \xi_{t+1}^{-\gamma}} \tag{44}$$

- If $x$ is log-normal distributed with , we have

$$E \exp(x) = \exp^{Ex + \frac{1}{2} var(\log \xi)} \tag{45}$$

# Equity Premium

- We assume $E\xi = 1$, so we have

$$E\xi = E\exp\{\log(\xi)\} --> E\log(\xi) = -\frac{1}{2}var(\log\xi) \equiv -\frac{1}{2}\sigma^2 \tag{46}$$

- We can use it to determine the interest rates:

$$\begin{aligned} R_{ft} &= \frac{1}{\beta E\exp^{-\gamma\log(\xi)}} \\ &= \frac{1}{\beta}\frac{1}{\exp^{-\gamma E\log(\xi)+\frac{1}{2}\gamma^2\sigma^2}} \\ &= \frac{1}{\beta}\frac{1}{\exp^{\frac{\gamma}{2}\sigma^2+\frac{1}{2}\gamma^2\sigma^2}} \end{aligned} \tag{47}$$

- or we have

$$\log R_{ft} = -\log(\beta) - (\frac{\gamma}{2}\sigma^2 + \frac{1}{2}\gamma^2\sigma^2) \tag{48}$$

# Equity Premium

- And the expect return

$$\begin{aligned} E_t \frac{Q_{t+1} + D_{t+1}}{Q_t} &= \frac{\kappa + 1}{\kappa} = E_t R_{et+1} \\ &= \frac{1}{\beta E_t \xi_{t+1}^{1-\gamma}} = \frac{1}{\beta} \frac{1}{\exp^{-\frac{1-\gamma}{2}\sigma^2 + \frac{1}{2}(\gamma-1)^2\sigma^2}} \end{aligned} \tag{49}$$

- so

$$\log E_t R_{et+1} = -\log(\beta) - \left(-\frac{1-\gamma}{2}\sigma^2 + \frac{1}{2}(\gamma - 1)^2\sigma^2\right) \tag{50}$$

- and the risk premium

$$\begin{aligned} & \log E_t R_{et+1} - \log R_{ft} \\ = & -\left(-\frac{1-\gamma}{2}\sigma^2 + \frac{1}{2}(\gamma-1)^2\sigma^2\right) + \left(\frac{\gamma}{2}\sigma^2 + \frac{1}{2}\gamma^2\sigma^2\right) \\ = & \gamma\sigma^2 \end{aligned} \tag{51}$$

# Equity Premium Puzzle

- In the data, the consumption growth volatility is 0.5508%,namely $\sigma = 0.5508\%$. The risk premium yield by the model is

$$(\log E_t R_{et+1} - \log R_{ft}) * 400 = 400\gamma * \sigma^2 \tag{52}$$

- The risk premium in the date is 7.23% per year.
- We generate the following table

Table 1. Risk Premium

| | $\gamma = 1$ | $\gamma = 2$ | $\gamma = 10$ |
|---|---|---|---|
| $E[R_e - R_f]$ | 0.0121% | 0.0243% | 0.1214% |

# Equity Premium Puzzle

- We see that the equity premium is given by

$$E[R_e - R_f] = \gamma\sigma^2 \tag{53}$$

- We now check the covariance between consumption growth rate and the risk return:

$$cov(\log(\frac{C_{t+1}}{D_{t+1}}), \log(\frac{Q_{t+1} + D_{t+1}}{D_t})) = \sigma^2 \tag{54}$$

- so we have

$$E[R_e - R_f] = \gamma cov(g_{t+1}, r_{e,t+1}) \tag{55}$$

- We now show the above results holds for general case.

# Equity Premium Puzzle

- For the general case, $C_t \neq D_t$

$$\begin{aligned} 1 &= \beta E_t \frac{C_t^{\gamma}}{C_{t+1}^{\gamma}} R_{et+1} \\ &= \beta E_t \exp(-\gamma g_{t+1} + r_{t+1}) \end{aligned} \tag{56}$$

$$\begin{aligned} 1 &= \beta R_{ft} E_t \frac{C_t^{\gamma}}{C_{t+1}^{\gamma}} \\ &= \beta E_t \exp(-\gamma g_{t+1} + r_{ft}) \end{aligned} \tag{57}$$

# Equity Premium Puzzle

- We do a second order approximation. Define $x_{t+1} - \bar{x}_{t+1} = \hat{x}_{t+1}$, where $\bar{x}_{t+1} = E_t x_{t+1}$

$$\begin{aligned} 1 &= \beta \exp(-\gamma \bar{g}_{t+1} + \bar{r}_{t+1}) E_t \exp(-\gamma \hat{g}_{t+1} + \hat{r}_{t+1}) \qquad (58) \\ &= \beta \exp(-\gamma \bar{g}_{t+1} + \bar{r}_{t+1}) \times \\ & E_t[1 - \gamma \hat{g}_{t+1} + \hat{r}_{t+1} + \frac{1}{2}(\gamma^2 \hat{g}_{t+1}^2 + \hat{r}_{t+1}^2 - 2\gamma \hat{g}_{t+1}\hat{r}_{t+1})] \\ &= \beta \exp(-\gamma \bar{g}_{t+1} + \bar{r}_{t+1}) E_t[1 + \frac{1}{2}(\gamma^2 \hat{g}_{t+1}^2 + \hat{r}_{t+1}^2 - 2\gamma \hat{g}_{t+1}\hat{r}_{t+1})] \end{aligned}$$

- or we have

$$-\gamma \bar{g}_{t+1} + \bar{r}_{t+1} = -\log(\beta) - \frac{1}{2} E_t(\gamma^2 \hat{g}_{t+1}^2 + \hat{r}_{t+1}^2 - 2\gamma \hat{g}_{t+1}\hat{r}_{t+1}) \qquad (59)$$

- similary we have

$$-\gamma \bar{g}_{t+1} + \bar{r}_{ft} = -\log(\beta) - \frac{1}{2} E_t(\gamma^2 \hat{g}_{t+1}^2)$$

# Equity Premium Puzzle

- or we have

$$-\gamma\bar{g}_{t+1} + \bar{r}_{t+1} = -\log(\beta) - \frac{1}{2}E_t(\gamma^2\hat{g}_{t+1}^2 + \hat{r}_{t+1}^2 - 2\gamma\hat{g}_{t+1}\hat{r}_{t+1}) \tag{60}$$

- similary we have

$$-\gamma\bar{g}_{t+1} + \bar{r}_{ft} = -\log(\beta) - \frac{1}{2}E_t(\gamma^2\hat{g}_{t+1}^2) \tag{61}$$

- the premium is

$$\begin{aligned}\bar{r}_{t+1} - \bar{r}_{ft} &= \gamma E_t\hat{g}_{t+1}\hat{r}_{t+1} - \frac{1}{2}E_t\hat{r}_{t+1}^2 \\ &= \gamma cov_t(g_{t+1}, r_{t+1}) - \frac{1}{2}var(r_{t+1}) \end{aligned} \tag{62}$$

# Equity Premium Puzzle

| Country | Sample period | $\overline{aer_e}$ | $\sigma(er_e)$ | $\sigma(m)$ | $\sigma(\Delta c)$ | $\rho(er_e, \Delta c)$ | $\text{cov}(er_e, \Delta c)$ | RRA(1) | RRA(2) |
|---|---|---|---|---|---|---|---|---|---|
| USA | 1947.2–1998.3 | 8.071 | 15.271 | 52.853 | 1.071 | 0.205 | 3.354 | 240.647 | 49.326 |
| AUL | 1970.1–1998.4 | 3.885 | 22.403 | 17.342 | 2.059 | 0.144 | 6.640 | 58.511 | 8.421 |
| CAN | 1970.1–1999.1 | 3.968 | 17.266 | 22.979 | 1.920 | 0.202 | 6.694 | 59.266 | 11.966 |
| FR | 1973.2–1998.3 | 8.308 | 23.175 | 35.848 | 2.922 | −0.093 | −6.315 | $< 0$ | 12.270 |
| GER | 1978.4–1997.3 | 8.669 | 20.196 | 42.922 | 2.447 | 0.029 | 1.446 | 599.468 | 17.542 |
| ITA | 1971.2–1998.1 | 4.687 | 27.068 | 17.314 | 1.665 | −0.006 | −0.252 | $< 0$ | 10.400 |
| JAP | 1970.2–1998.4 | 5.098 | 21.498 | 23.715 | 2.561 | 0.112 | 6.171 | 82.620 | 9.260 |
| NTH | 1977.2–1998.3 | 11.421 | 16.901 | 67.576 | 2.510 | 0.032 | 1.344 | 849.991 | 26.918 |
| SWD | 1970.1–1999.2 | 11.539 | 23.518 | 49.066 | 1.851 | 0.015 | 0.674 | 1713.197 | 26.501 |
| SWT | 1982.2–1998.4 | 14.898 | 21.878 | 68.098 | 2.123 | −0.112 | −5.181 | $< 0$ | 32.076 |
| UK | 1970.1–1999.1 | 9.169 | 21.198 | 43.253 | 2.511 | 0.093 | 4.930 | 185.977 | 17.222 |
| USA | 1970.1–1998.3 | 6.353 | 16.976 | 37.425 | 0.909 | 0.274 | 4.233 | 150.100 | 41.178 |
| SWD | 1920–1997 | 6.540 | 18.763 | 34.855 | 5.622 | 0.167 | 8.830 | 74.062 | 12.400 |
| UK | 1919–1997 | 8.674 | 21.277 | 40.767 | 5.630 | 0.351 | 21.042 | 41.223 | 14.483 |
| USA | 1891–1997 | 6.723 | 18.496 | 36.345 | 6.437 | 0.495 | 29.450 | 22.827 | 11.293 |

# Risk free rate puzzle

- In order to solve the equity premium puzzle, we need incredible high risk aversion coefficient!!! But it creat another problem
- Risk free rate is equal to

$$\bar{r}_{ft} = -\log(\beta) + \gamma \bar{g}_{t+1} - \frac{\gamma^2}{2}\sigma_g^2 \tag{63}$$

- High $\gamma$ would leads high risk free rate. One need high $\beta$ to the match the low risk free rate. This is called as risk free rate puzzle.

# Risk free rate puzzle

| Country | Sample period | $\overline{r_f}$ | $\overline{\Delta c}$ | $\sigma(\Delta c)$ | RRA(1) | TPR(1) | RRA(2) | TPR(2) |
|---|---|---|---|---|---|---|---|---|
| USA | 1947.2–1998.3 | 0.896 | 1.951 | 1.071 | 240.647 | −136.270 | 49.326 | −81.393 |
| AUL | 1970.1–1998.4 | 2.054 | 2.071 | 2.059 | 58.511 | −46.512 | 8.421 | −13.880 |
| CAN | 1970.1–1999.1 | 2.713 | 2.170 | 1.920 | 59.266 | −61.154 | 11.966 | −20.618 |
| FR | 1973.2–1998.3 | 2.715 | 1.212 | 2.922 | $< 0$ | N/A | 12.270 | −5.735 |
| GER | 1978.4–1997.3 | 3.219 | 1.673 | 2.447 | 599.468 | 9757.265 | 17.542 | −16.910 |
| ITA | 1971.2–1998.1 | 2.371 | 2.273 | 1.665 | $< 0$ | N/A | 10.400 | −19.765 |
| JAP | 1970.2–1998.4 | 1.388 | 3.233 | 2.561 | 82.620 | −41.841 | 9.260 | −25.735 |
| NTH | 1977.2–1998.3 | 3.377 | 1.671 | 2.510 | 849.991 | 21349.249 | 26.918 | −18.769 |
| SWD | 1970.1–1999.2 | 1.995 | 1.001 | 1.851 | 1713.197 | 48590.956 | 26.501 | −12.506 |
| SWT | 1982.2–1998.4 | 1.393 | 0.559 | 2.123 | $< 0$ | N/A | 32.076 | 6.636 |
| UK | 1970.1–1999.1 | 1.301 | 2.235 | 2.511 | 185.977 | 676.439 | 17.222 | −27.838 |
| USA | 1970.1–1998.3 | 1.494 | 1.802 | 0.909 | 150.100 | −175.916 | 41.178 | −65.701 |
| SWD | 1920–1997 | 2.209 | 1.730 | 2.811 | 74.062 | 90.793 | 12.400 | −13.165 |
| UK | 1919–1997 | 1.255 | 1.472 | 2.815 | 41.223 | 7.913 | 14.483 | −11.749 |
| USA | 1891–1997 | 2.020 | 1.760 | 3.218 | 22.827 | −11.162 | 11.293 | −11.247 |

- The utility function is

$$U = \sum \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \tag{64}$$

- The household problem is to maximize his utility with the constraint:

$$\sum \beta^t q_t C_t \leq W \tag{65}$$

  where $W$ is his life-time income.

- The intertemporal elascticity of substitutioncan be obtained from the following equation

$$\frac{C_t^{-\gamma}}{q_t} = \frac{C_{t+j}^{-\gamma}}{q_{t+j}} \tag{66}$$

- So we have

$$(C_{t+j}/C_t)^{-\gamma} = q_{t+j}/q_t \tag{67}$$

- The relative consumption is inverse function of the relative price. The intertemporal elascticity of substitution is obtained by

$$IES = -\frac{d\ln(C_{t+j}/C_t)}{d\ln(q_{t+j}/q_t)} = \frac{1}{\gamma} \tag{68}$$

- To consider the risk aversion. We look at the one period utility function

$$U = \frac{C_t^{1-\gamma}}{1-\gamma} \tag{69}$$

- Consider a certainty consumption $\mu C_t$ and an uncertainty consumption $\lambda_t C_t$ which are equivalent to the household. We then must have

$$E\frac{\lambda_t^{1-\gamma} C_t^{1-\gamma}}{1-\gamma} = \frac{C_t^{1-\gamma}}{1-\gamma} \tag{70}$$

- or

$$E\lambda_t^{1-\gamma} = 1 \tag{71}$$

- We need to solve
$$E\lambda_t^{1-\gamma} = 1 \tag{72}$$
- Approximiate its around $\bar{\lambda} = E\lambda_t$. We have
$$\lambda_t^{1-\gamma} = \bar{\lambda}^{1-\gamma} + (1-\gamma)\bar{\lambda}_t^{-\gamma}[\lambda_t - \bar{\lambda}] - \frac{(1-\gamma)\gamma}{2}\bar{\lambda}_t^{-\gamma-1}[\lambda_t - \bar{\lambda}]^2 \tag{73}$$
- or we have
$$1 = \bar{\lambda}^{1-\gamma} - \frac{(1-\gamma)\gamma}{2}\bar{\lambda}_t^{1-\gamma}E_t[\frac{\lambda_t - \bar{\lambda}}{\bar{\lambda}}]^2 \tag{74}$$

# The seperable utility

- or

$$\bar{\lambda} = \left[ \frac{1}{1 - \frac{(1-\gamma)\lambda}{2}\sigma^2} \right]^{\frac{1}{1-\gamma}} \tag{75}$$

- so

$$\bar{\lambda} = 1 + \frac{\gamma}{2}\sigma^2 \tag{76}$$

- the premium is

$$\bar{\lambda} - 1 = \frac{\gamma}{2}\sigma^2 \tag{77}$$

- a high risk aversion coefficient $\gamma$ require a high premium.

# The seperable utility

- Now consider the general utility function

$$U = \sum \beta^t u(C_t) \tag{78}$$

- Similary we have

$$\frac{u'(C_t)}{p_t} = \frac{u'(C_{t+j})}{p_{t+j}} \tag{79}$$

- This implies

$$\frac{u'(C_{t+j})}{u'(C_t)} = \frac{p_{t+j}}{p_t} \tag{80}$$

- Consider small price change $p_{t+j} = p_t + \Delta p_t$

## The seperable utility

- We approximiate $u'(C_{t+j})$, we have

$$u'(C_{t+j}) = u'(C_t) + u''(C_t)C_t\left(\frac{C_{t+j} - C_t}{C_t}\right) \tag{81}$$

$$\frac{u'(C_{t+j})}{u'(C_t)} = \frac{p_{t+j}}{p_t} \tag{82}$$

- so we have

$$1 + \frac{u''(C_t)C_t}{u'(C_t)}\left(\frac{C_{t+j} - C_t}{C_t}\right) = 1 + \frac{\Delta p_t}{p_t} \tag{83}$$

- or we have

$$\left(\frac{\Delta C_t}{C_t}/\frac{\Delta p_t}{p_t}\right) = \frac{1}{\frac{u''(C_t)C_t}{u'(C_t)}} = IES$$

# The seperable utility

- To derive the risk aversion coefficient, we define

$$E_t u(\lambda_t C_t) = u(C_t) \tag{84}$$

- Again approximate $u(\lambda_t C_t)$ around $U(C_t)$, we have

$$u(\lambda_t C_t) = u(C_t) + u'(C_t)[\lambda_t - 1]C_t + \frac{u''(C_t)}{2}[\lambda_t - 1]^2 C_t^2 \tag{85}$$

- or we have

$$E\lambda_t - 1 = -\frac{1}{2}\frac{u''(C_t)C_t}{u'(C_t)}E_t[\lambda_t - 1]^2 \tag{86}$$

- Notice $var(\lambda_t - 1) = E[\lambda_t - 1]^2 - (E\lambda_t - 1)^2$, the above equation suggest

$$var(\lambda_t - 1) = \sigma^2 = E[\lambda_t - 1]^2 + o(\sigma^3) \tag{87}$$

- So the premium depends on $\frac{u''(C_t)C_t}{u'(C_t)}$

# The seperable utility

- Generally speaking , the risk aversion coefficient and the intertemporal substitution coefficients in the time seperable utility case are caputured by the same parameters.
- However intertemporal substitution and risk aversion are two different concepts. So it is not clear that these should be the same.

# E-Z utility function

- Utility function

$$U_t = \left[c_t^\rho + \beta(E_t U_{t+1}^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{88}$$

- If $\rho = \alpha$. The utility function becomes

$$U_t^\rho = c_t^\rho + \beta(E_t U_{t+1}^\rho) \tag{89}$$

- If we iterate forward, we have

$$U_t^\rho = E_t \sum \beta^j c_{t+j}^\rho \tag{90}$$

  which is the usual utility function.

- If there is no uncertainty, again we have

$$U_t^\rho = \sum \beta^j c_{t+j}^\rho \tag{91}$$

  so $\rho$ captures the intertemporal substitution. And $\alpha$ captures the risk attitudes.

# Household problem

- Utility function

$$U_t = \left[ c_t^{\rho} + \beta (E_t U_{t+1}^{\alpha})^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}} \quad (92)$$

- The budget constraint:

$$c_t + \sum_{i=1}^{N} S_{i,t+1} = \sum_{i=1}^{N} S_{t,i} R_{i,t} \quad (93)$$

- Define the total asset as

$$A_t = \sum_{i=1}^{N} S_{t,i} R_{i,t} \quad (94)$$

# Household problem

- The net asset in the next period is then

$$
\begin{aligned}
A_{t+1} &= \sum_{i=1}^{N} S_{t+1,i} R_{t+1,i} \\
&= \sum_{i=1}^{N} S_{i,t+1} \left[ \sum_{i=1}^{N} \omega_{t,i} R_{t,i} \right] \\
&= (A_t - c_t) \omega_t^{'} R_t \\
&= (A_t - c_t) M_t
\end{aligned}
$$

- Where $M_t$ is the total portfolio return. Take $M_t$ as given, we have :

$$A_{t+1} = (A_t - c_t) M_{t+1} \tag{95}$$

# Household problem

- We can solve the above problem by guess and verification strategy. Guess the value function is linear in wealth. Namely

$$U_t = \phi_t A_t \tag{96}$$

- where $\phi_t$ remains to be determine. With the above guess, we have

$$\max U_t = \{c_t^\rho + \beta(A_t - c_t)^\rho [E_t \phi_{t+1}^\alpha m_{t+1}^\alpha]^{\frac{\rho}{\alpha}}\}^{\frac{1}{\rho}} \tag{97}$$

- Define $\mu^* = [E_t \phi_{t+1}^\alpha m_{t+1}^\alpha]^{\frac{1}{\alpha}}$ We then have the foc with respect to consumption is

# Household problem

- Define $\mu^* = \left[E_t \phi_{t+1}^{\alpha} M_{t+1}^{\alpha}\right]^{\frac{1}{\alpha}}$ We then have the foc with respect to consumption is

$$\max_{c_t} U_t = \left\{c_t^{\rho} + \beta(A_t - c_t)^{\rho} \mu^{*\rho}\right\}^{\frac{1}{\rho}} \tag{98}$$

- or it yields

$$c_t^{\rho-1} = \beta(A_t - c_t)^{\rho-1} \mu^{*\rho} \tag{99}$$

- we can then solve $\mu^{*\rho}$

# Household problem

- We have obtained

$$c_t^{\rho-1} = \beta(A_t - c_t)^{\rho-1}\mu^{*\rho} \tag{100}$$

- so we have

$$\begin{aligned} U_t &= \{c_t^\rho + \beta(A_t - c_t)^\rho \frac{c_t^{\rho-1}}{\beta(A_t - c_t)^{\rho-1}}\}^{\frac{1}{\rho}} \\ &= \{c_t^\rho + (A_t - c_t)c_t^{\rho-1}\}^{\frac{1}{\rho}} \\ &= \phi_t A_t \end{aligned} \tag{101}$$

- or

$$\phi_t = \left(\frac{c_t}{A_t}\right)^{\frac{\rho-1}{\rho}} \tag{102}$$

# Household problem

- By defininition we have

$$\mu^{*\rho} = \left[E\left(\frac{c_{t+1}}{A_{t+1}}\right)^{\frac{\rho-1}{\rho}\alpha} m_{t+1}^{\alpha}\right]^{\frac{\rho}{\alpha}} \tag{103}$$

$$= (A_t - c_t)^{1-\rho}\left[E\left(\frac{c_{t+1}}{m_{t+1}}\right)^{\frac{\rho-1}{\rho}\alpha} m_{t+1}^{\alpha}\right]^{\frac{\rho}{\alpha}}$$

- And by

$$c_t^{\rho-1} = \beta(A_t - c_t)^{\rho-1}\mu^{*\rho} \tag{104}$$

- We obtain

$$c_t^{(\rho-1)\frac{\alpha}{\rho}} = \beta^{\frac{\alpha}{\rho}}\left[E\left(\frac{c_{t+1}}{m_{t+1}}\right)^{\frac{\rho-1}{\rho}\alpha} m_{t+1}^{\alpha}\right] \tag{105}$$

# Household problem

- Let $\theta = \frac{\alpha}{\rho}$, we then have

$$E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho-1} m_{t+1} \right]^{\theta} = 1 \tag{106}$$

- So far we have obtain the Euler equation for consumption. Now we try to determine the portfolio. Given $c_t^*$, the problem is to solve

$$\max_{\omega_{i,t+1}} \left\{ E_t \phi_{t+1}^{\alpha} \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^{\alpha} \right\}^{\frac{1}{\alpha}} \tag{107}$$

- with the constraint

$$\sum_{i=1}^{N} \omega_{i,t+1} = 1 \tag{108}$$

# Household problem

- Again this is a static problem: The first order conditions are

$$\begin{aligned} & \left\{ E_t \phi_{t+1}^{\alpha} \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^{\alpha} \right\}^{\frac{1}{\alpha}-1} \times \\ & E_t \phi_{t+1}^{\alpha} \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^{\alpha-1} R_{i,t+1} \\ = & \lambda_t \end{aligned} \tag{109}$$

- Mutiply this equation by $\omega_{i,t+1}$,sum over $i$ to derive :

$$\left\{ E_t \phi_{t+1}^{\alpha} \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^{\alpha} \right\}^{\frac{1}{\alpha}} = \lambda_t \tag{110}$$

# Household problem

- This yields

$$
\begin{aligned}
& E_t\phi_{t+1}^{\alpha}\left(\sum_{i=1}^{N}\omega_{i,t+1}R_{i,t+1}\right)^{\alpha-1} R_{i,t+1} \\
= \quad & E_t\phi_{t+1}^{\alpha}\left(\sum_{i=1}^{N}\omega_{i,t+1}R_{i,t+1}\right)^{\alpha} \qquad (111)
\end{aligned}
$$

- by definition

$$
m_{t+1} = \sum_{i=1}^{N}\omega_{i,t+1}R_{i,t+1} \qquad (112)
$$

# Household problem

- We now substitute $\phi_{t+1}$ out by

$$\phi_t = \left(\frac{c_t}{A_t}\right)^{\frac{\rho-1}{\rho}} \tag{113}$$

- or we have

$$E_t\left(\frac{c_{t+1}}{A_{t+1}}\right)^{\theta(\rho-1)} m_{t+1}^{\alpha-1} R_{i,t+1} = E_t\left(\frac{c_{t+1}}{A_{t+1}}\right)^{\theta(\rho-1)} m_{t+1}^{\alpha} \tag{114}$$

- finally we have

$$A_{t+1} = (A_t - c_t) m_{t+1} \tag{115}$$

# Household problem

- Further simplication yields

$$E_t \left(\frac{c_{t+1}}{m_{t+1}}\right)^{\theta(\rho-1)} m_{t+1}^{\alpha-1} R_{i,t+1} = E_t \left(\frac{c_{t+1}}{m_{t+1}}\right)^{\theta(\rho-1)} m_{t+1}^{\alpha}$$
$$= E_t c_{t+1}^{\theta(\rho-1)} m_{t+1}^{\theta} \qquad (116)$$

- Finally use that

$$E_t \left[\beta \left(\frac{c_{t+1}}{c_t}\right)^{\rho-1} m_{t+1}\right]^{\theta} = 1 \qquad (117)$$

- multiplying both side by $c_t^{-\theta(\rho-1)} \beta^{\theta}$ we yields

$$E_t \beta^{\theta} \left(\frac{c_{t+1}}{c_t}\right)^{\theta(\rho-1)} m_{t+1}^{\theta-1} R_{i,t+1} = 1 \qquad (118)$$

# Household problem

- Consider two asset. One is the stock, the other is risk free bonds. The bond has zero supply. So in equilibrium we have

$$\omega_{b,t+1} = 0 \tag{119}$$

and

$$\omega_{et+1} = 1 \tag{120}$$

where $\omega_{b,t+1}$ is the weight on bond and $\omega_{et+1}$ is the weight on stock

- hence we have

$$m_{t+1} = R_{et+1} \tag{121}$$

# Risk Premium

- We hence has

$$E_t\beta^\theta\left(\frac{c_{t+1}}{c_t}\right)^{\theta(\rho-1)}R^{\theta}_{et+1}=1 \tag{122}$$

- and

$$E_t\beta^\theta\left(\frac{c_{t+1}}{c_t}\right)^{\theta(\rho-1)}R^{\theta-1}_{et+1}R_{ft}=1 \tag{123}$$

- We can rewrite them in expoential terms

$$\beta^\theta E_t e^{\theta(\rho-1)\Delta c_{t+1}+(\theta-1)\log m_{t+1}+r_{i,t+1}}=1 \tag{124}$$

# Risk Premium

- Assume normal distribution for $\Delta c_{t+1}$, $\log m_{t+1}$, $r_{it+1}$, we hence has

$$\begin{aligned} & \theta \log \beta + \theta(\rho - 1) E\Delta c_{t+1} + (\theta - 1) Em_{t+1} + Er_{i,t+1} \\ +\frac{1}{2}\sigma^2 &= 0 \end{aligned} \quad (125)$$

- where

$$\begin{aligned} \sigma^2 &= \theta^2(\rho-1)^2\sigma_c^2 + (\theta-1)^2\sigma_m^2 + \sigma_i^2 + \quad (126) \\ & 2\theta(\rho-1)\sigma_{c,i} + 2\theta(\rho-1)(\theta-1)\sigma_{c,m} + 2(\theta-1)\sigma_{i,m} \end{aligned}$$

  where $\sigma_{c,i}$ is the covariance btween consumption growth and the log return.

- For risk free return, we have $\sigma_f^2 = 0; \sigma_{c,f} = 0; \sigma_{i,m} = 0.$

# Risk Premium

- Hence the risk premium is

$$Er_{t+1} - r_{f,t+1} \simeq \theta(1-\rho)cov(\Delta c_{t+1}, r_{et+1}) + (1-\theta)\sigma_e^2 - \frac{1}{2}\sigma_e^2 \quad (127)$$

- in the case $\rho = \alpha$, so $\theta = 1$, this reduces to

$$Er_{t+1} - r_{f,t+1} \simeq (1-\rho)cov(\Delta c_{t+1}, r_{et+1}) - \frac{1}{2}\sigma_e^2 \quad (128)$$

  which is the same as before.
- We requires a very large $\gamma$

$$\gamma = 1 - \rho \quad (129)$$

- and in this case, we can explain the risk premium puzzle but generate risk free rate puzzle.

# Risk Premium

- The E-Z preference can solve both puzzle. To see this. Let $\rho = 1$. And in such case, the risk free rate is given by

$$Er_{t+1} - r_{f,t+1} \simeq \theta(1-\rho)cov(\Delta c_{t+1}, r_{et+1}) + (1-\theta)\sigma_e^2 - \frac{1}{2}\sigma_e^2 \quad (130)$$

$$r_{ft} = -\theta \log \beta + (1-\theta)E_t r_{et+1} - \frac{1}{2}[(\theta-1)^2\sigma_e^2] \quad (131)$$

- Combine the above two equations we have

$$\theta r_{ft} + (\theta-1)[(1-\theta)\sigma_e^2 - \frac{1}{2}\sigma_e^2] = -\theta \log \beta - \frac{1}{2}[(\theta-1)^2\sigma_e^2] \quad (132)$$

$$\begin{aligned} \theta r_{ft} &= -\theta \log \beta - \frac{1}{2}[(\theta-1)^2 + 2(\theta-1)(1-\theta) - (\theta-1)]\sigma_e^2 \quad (133) \\ &= -\theta \log \beta + \frac{1}{2}[(\theta-1)\theta]\sigma_e^2 \end{aligned}$$

- or we have

$$r_{ft} = -\log \beta + \frac{1}{2}(\theta-1)\sigma_e^2 < -\log \beta \quad (134)$$

- The representative household solves:

$$\max \ U_t = \left\{ \left[ c_t^{\theta} (1 - n_t)^{1-\theta} \right]^{\rho} + \beta \left( E_t U_{t+1}^{\gamma} \right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho}}$$

- s. t

$$c_t + k_{t+1} = A_t k_t^{\alpha} n_t^{1-\alpha} + (1 - \delta) k_t$$

# EZ-in RBC

- Set-up the Lagrangian function

$$
\begin{aligned}
U(k_t, A_t) &= \left\{ \left[ c_t^{\theta} (1-n_t)^{1-\theta} \right]^{\rho} + \beta \left( E_t U_{t+1}^{\gamma} \right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho}} \\
&\quad + \lambda_t \left[ A_t k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t - c_t - k_{t+1} \right]
\end{aligned} \tag{135}
$$

- The first oder conditions with respect to $c_t, n_t$ and $k_{t+1}$ are :

$$
\begin{aligned}
& \left\{ \left[ c_t^{\theta} (1-n_t)^{1-\theta} \right]^{\rho} + \beta \left( E_t U_{t+1}^{\gamma} \right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho}-1} \\
& \times \left[ c_t^{\theta} (1-n_t)^{1-\theta} \right]^{\rho-1} \theta c_t^{\theta-1} \\
= \; & \lambda_t
\end{aligned} \tag{136}
$$

# EZ-in RBC

- With respect to labor is

$$\begin{aligned} & \left\{ \left[ c_t^{\theta} (1 - n_t)^{1-\theta} \right]^{\rho} + \beta \left( E_t U_{t+1}^{\gamma} \right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho} - 1} \\ & \times \left[ c_t^{\theta} (1 - n_t)^{1-\theta} \right]^{\rho - 1} (1 - \theta) n_t^{\theta - 1} \\ = \; & \lambda_t \frac{(1 - \alpha) y_t}{n_t} \end{aligned} \tag{137}$$

- and with respect to capital is

$$\begin{aligned} & \left\{ \left[ c_t^{\theta} (1 - n_t)^{1-\theta} \right]^{\rho} + \beta \left( E_t U_{t+1}^{\gamma} \right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho} - 1} \\ & \times \beta \left( E_t U_{t+1}^{\gamma} \right)^{\frac{\rho}{\gamma} - 1} \\ & \times E U_{t+1}^{\gamma - 1} \frac{\partial U'}{\partial k'} \\ = \; & \lambda_t \end{aligned}$$

# EZ-in RBC

- Envelop theory

$$\frac{\partial U(k_t, A_t)}{\partial k_t} = \lambda_t[\alpha \frac{y_t}{k_t} + (1-\delta)] \tag{138}$$

- So the Euler equation is

$$\begin{aligned} & U_t^{1-\rho} \left[ c_t^{\theta} (1-n_t)^{1-\theta} \right]^{\rho-1} \theta c_t^{\theta-1} \\ = \quad & U_t^{1-\rho} \beta \left( E_t U_{t+1}^{\gamma} \right)^{\frac{\rho}{\gamma}-1} \\ & \times E_t U_{t+1}^{\gamma-\rho} \left[ c_{t+1}^{\theta} (1-n_{t+1})^{1-\theta} \right]^{\rho-1} \theta c_{t+1}^{\theta-1} \end{aligned} \tag{139}$$

- it can be reduced further to

$$\begin{aligned} \left[ c_t^{\theta} (1-n_t)^{1-\theta} \right]^{\rho-1} c_t^{\theta-1} \quad = \quad & \beta \left( E_t U_{t+1}^{\gamma} \right)^{\frac{\rho-\gamma}{\gamma}} \\ & \times E_t U_{t+1}^{\gamma-\rho} \left[ c_{t+1}^{\theta} (1-n_{t+1})^{1-\theta} \right]^{\rho-1} c_{t+1}^{\theta-1} \end{aligned} \tag{140}$$

# EZ-in RBC

in summary we have

$$\begin{aligned} & \left[c_t^{\theta}(1-n_t)^{1-\theta}\right]^{\rho-1} c_t^{\theta-1} \\ = \; & \beta \left(E_t U_{t+1}^{\gamma}\right)^{\frac{\rho-\gamma}{\gamma}} \times E_t U_{t+1}^{\gamma-\rho} \left[c_{t+1}^{\theta}(1-n_{t+1})^{1-\theta}\right]^{\rho-1} c_{t+1}^{\theta-1} \end{aligned} \tag{141}$$

$$\theta c_t^{\theta-1} \frac{(1-\alpha)y_t}{n_t} = (1-\theta) n_t^{-\theta} \tag{n}$$

$$c_t + k_{t+1} = A_t k_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t \tag{c}$$

$$U_t = \left\{ \left[c_t^{\theta}(1-n_t)^{1-\theta}\right]^{\rho} + \beta \left(E_t U_{t+1}^{\gamma}\right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho}} \tag{U}$$

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha} \tag{y}$$