Notes on Asset Prices in the exchange/production economy

Pengfei Wang

Hong Kong University of Science and Technology

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Real business cycle models are arguably successful at mimicking the cyclical behavior of macroeconomic quantities.

However, Mehra and Prescott (1985) show that utility specifications common in RBC models have counterfactual implications for asset prices.

We investigate the performance of the RBC model in mimicking the difference between the average return to stocks and bonds, or the risk premium.

We starts with an endowment economy and then we move to the production economy.
An continuum of identical households with time-separable utility function

\[ U = E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \]  

(1)

Two assets. One asset is bond, the other asset is a tree.

The tree yields a fruit sequence \( D_t \) according to some distribution to be specified.

The tree can be traded among households. Its price is \( P_t \). The bonds return is \( R_{ft} \) between period \( t \) to period \( t+1 \).
The Household Problem

- The household maximize

\[ E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \]  

(2)

- with a consequence of constraint

\[ C_t + Q_t s_{t+1} + \frac{B_{t+1}}{R_{ft}} = (Q_t + D_t)s_t + B_t \]  

(3)

- where \( B_t \) is the bond holding in the beginning of period and \( s_t \) is the stock holding.
The Bellman Equation

- Notice the household has two state variables, $s_t, B_t$
- Three variables need to be solved, $C_t, s_{t+1}, B_{t+1}$
- Also the price $Q_t, R_{ft}, D_t$ are also exogenous states to the households. These are aggregate states. We denote them $\Phi_t$.
- Define $V(s_t, B_t; \Phi_t)$ as $V_t(s_t, B_t)$ as the value function, we have
The Bellman Equation

Define $V(s_t, B_t; \Phi_t)$ as $V_t(s_t, B_t)$ as the value function, we have

$$V_t(s_t, B_t) = \max_{s_{t+1}, B_{t+1}, C_t} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \beta E_t V_{t+1}(s_{t+1}, B_{t+1}) 
+ \lambda_t [(Q_t + D_t)s_t + B_t - C_t - Q_ts_{t+1} - \frac{B_{t+1}}{R_{ft}}] \right\} \tag{4}$$

FOC yields

$$C_t^{-\gamma} = \lambda_t \tag{5}$$

$$\lambda_t Q_t = \beta E_t \frac{\partial V_{t+1}(s_{t+1}, B_{t+1})}{\partial s_{t+1}} \tag{6}$$

$$\lambda_t \frac{1}{R_{ft}} = \beta E_t \frac{\partial V_{t+1}(s_{t+1}, B_{t+1})}{\partial B_{t+1}} \tag{7}$$
The Bellman Equation

The Envelop theorem yields

$$\frac{\partial V_t(s_t, B_t)}{\partial B_t} = \lambda_t (Q_t + D_t)$$  \hspace{1cm} (8)

and

$$\frac{\partial V_t(s_t, B_t)}{\partial s_t} = \lambda_t$$  \hspace{1cm} (9)

so we have two Euler equations below

$$\lambda_t \frac{1}{R_{ft}} = \beta E_t \lambda_{t+1}$$  \hspace{1cm} (10)

and

$$\lambda_t Q_t = \beta E_t \lambda_{t+1}(Q_{t+1} + D_{t+1})$$  \hspace{1cm} (11)
Equilibrium

The equilibrium is a sequence of price and quantity
\{Q_t, R_{ft}, C_t, s_{t+1}, B_{t+1}\} such that given \{Q_t, R_{ft}, D_t\}, such that
\{C_t, s_{t+1}, B_{t+1}\} solves the individual problem and all market clears.

This implies

\[ s_{t+1} = 1 \]  \hspace{1cm} (12)

and

\[ B_{t+1} = 0 \]  \hspace{1cm} (13)

\[ C_t = D_t \]
So we have the price of the tree is given by

\[ D_t^{-\gamma} Q_t = \beta E_t D_{t+1}^{-\gamma} [Q_{t+1} + D_{t+1}] \]  \hspace{1cm} (14)

and the return is

\[ D_t^{-\gamma} = \beta \frac{1}{R_{ft}} E_t D_{t+1}^{-\gamma} \]  \hspace{1cm} (15)

The above two equations define

\[ Q_t = Q(D_t); \ R_{ft} = R_f(D_t) \]  \hspace{1cm} (16)

the prices as a function of dividend.
Equilibrium

- If $D_t$ follows a Markovian Process, with the transitional probability

$$\Pr(D_{t+1} = d_j | D_t = d_i) = \pi_{ij}$$  \hspace{1cm} (17)

- We then can calculate interest rate and the stock price by

$$d_i^{-\gamma} = \beta R_i \sum \pi_{ij} d_j^{-\gamma}$$  \hspace{1cm} (18)

- and the stock price as

$$d_i^{-\gamma} Q_i = \beta \sum \pi_{ij} d_j^{-\gamma} [Q_j + d_j]$$  \hspace{1cm} (19)
For an example, consider $d_i = [1, 2]$ and the and $\pi_{ij} = .5, \gamma = 1$, we have

$$R_1 = \frac{1}{\beta^\frac{1}{2} [1^{-1} + 2^{-1}]}$$  \tag{20}

and

$$R_2 = \frac{2^{-1}}{\beta^\frac{1}{2} [1^{-1} + 2^{-1}]}$$  \tag{21}

and

$$Q_1 = \frac{\beta}{2} [Q_1 + 1 + \frac{1}{2} (Q_2 + 2)]$$ \tag{22}$$

$$Q_2 \frac{1}{2} = \frac{\beta}{2} [Q_1 + 1 + \frac{1}{2} (Q_2 + 2)]$$ \tag{23}

Which can be used to solve $Q_1, Q_2$. 
Equilibrium

Generally the stock price equals to

\[ Q_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \]  \hspace{1cm} (24)

In deriving the above result, we have used the transversality condition such that

\[ E_t \lim_{j \to 0} \beta^j \lambda_{t+j} Q_{t+j} = 0 \]  \hspace{1cm} (25)
consider a sequence of $P_t$ which satisfy

$$P_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} P_{t+1}$$

We can verify that

$$\tilde{Q}_t = Q_t + P_t$$

$$\tilde{Q}_t = \beta E_t [\tilde{Q}_{t+1} + \tilde{D}_{t+1}]$$

However for any $P_t \neq 0$, the price $\tilde{Q}_t$ can not support the equilibrium. We call $Q_t$ as the fundamental price of the asset and $P_t$ as the bubble component.
To understand the results, we consider a deterministic case. Namely $D_t = \bar{D}$. In such case we

$$R_{ft} = \frac{1}{\beta};$$

$$Q_t = \frac{\beta}{1-\beta}D$$  \hspace{1cm} (29)

And the utility level for the household in the equilibrium path is

$$U = \frac{1}{1-\beta} u(D)$$  \hspace{1cm} (30)

Now consider $P_t = \frac{1}{\beta^t} P_0$ and the price $\tilde{Q}_t = \frac{\beta}{1-\beta} D + \frac{1}{\beta^t} P_0$. Clearly we have

$$\frac{\beta}{1-\beta} D + \frac{1}{\beta^t} P_0 = \beta \left[ \frac{\beta}{1-\beta} D + D + \frac{1}{\beta^{t+1}} P_0 \right]$$  \hspace{1cm} (31)

or

$$\tilde{Q}_t = \beta [\tilde{Q}_{t+1} + D_{t+1}]$$  \hspace{1cm} (32)
We have already seen the price $\tilde{Q}_t = \frac{\beta}{1-\beta} D + \frac{1}{\beta^t} P_0$ also satisfy the household’s Euler equation:

$$\tilde{Q}_t = \beta [\tilde{Q}_{t+1} + D_{t+1}]$$  \hspace{1cm} (33)

However transversality condition is violated

$$\lim_{j \to 0} \beta^j \lambda_{t+j} Q_{t+j} = u'(D) P_0 \neq 0 \hspace{1cm} (34)$$

for $P_0 \neq 0$.

We consider the case with $P_0 > 0$. 
We consider the case with $P_0 > 0$. We want to prove that $B_{t+1} = 0, s_{t+1} = 1$ is not an equilibrium.

Consider the household decide $B_{t+1} = Q, s_{t+1} = 0$, and $C_0 = D + P_0$. In such case, household enjoy utility level

$$C_0 = D + P_0$$  \hspace{1cm} (35)

$$C_1 = \left[\frac{1}{\beta} - 1\right]Q = \frac{\beta}{1-\beta}D \times \frac{1-\beta}{\beta} = D$$  \hspace{1cm} (36)

$$C_t = D \text{ for } t \geq 1$$  \hspace{1cm} (37)
so we have

\[ \tilde{U} > \frac{1}{1 - \beta} u(D) \]  \hspace{1cm} (38)

so everyone would want to sell the stock in period 0. So for any \( P_0 > 0 \) can not be supported by the equilibrium.

Generally speaking, the only price for the stock is

\[ Q_t = E_t \sum \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \]  \hspace{1cm} (39)

So there can not be bubbles.
Now we consider the equity premium in the model. To obtain a closed form solution, we assume that \( \log D_t \) follows a random walk. Namely

\[
D_{t+1} = D_t \xi_t
\]

(40)

In this case,

\[
Q_tD_t^{-\gamma} = \beta E_tD_{t+1}^{-\gamma}[Q_{t+1} + D_{t+1}]
\]

(41)

Guess a solution such that \( Q_t / D_t = \kappa \), we can solve

\[
\kappa D_t^{1-\gamma} = \beta (\kappa + 1) E_tD_{t+1}^{1-\gamma}
\]

(42)

or we have

\[
\frac{\kappa + 1}{\kappa} = \frac{1}{\beta E_t \bar{\xi}_{t+1}^{1-\gamma}}
\]
The risk free rate is

$$D_t^{-\gamma} = \beta R_{ft} E_t D_{t+1}^{-\gamma}$$  \hspace{1cm} (43)

so we have

$$R_{ft} = \frac{1}{\beta E_t \xi_{t+1}^{-\gamma}}$$  \hspace{1cm} (44)

If $x$ is log-normal distributed with, we have

$$E \exp(x) = \exp^{E x + \frac{1}{2} \text{var}(\log \xi)}$$  \hspace{1cm} (45)
Equity Premium

- We assume $E\xi = 1$, so we have

$$E\xi = E \exp\{\log(\xi)\} \quad \rightarrow \quad E \log(\xi) = -\frac{1}{2} \text{var}(\log \xi) \equiv -\frac{1}{2} \sigma^2$$

(46)

- We can use it to determine the interest rates:

$$R_{ft} = \frac{1}{\beta E \exp^{-\gamma \log(\xi)}}$$

$$= \frac{1}{\beta} \frac{1}{\exp^{-\gamma E \log(\xi) + \frac{1}{2} \gamma^2 \sigma^2}}$$

$$= \frac{1}{\beta} \frac{1}{\exp^{\frac{\gamma}{2} \sigma^2 + \frac{1}{2} \gamma^2 \sigma^2}}$$

(47)

- or we have

$$\log R_{ft} = -\log(\beta) - \left(\frac{\gamma}{2} \sigma^2 + \frac{1}{2} \gamma^2 \sigma^2\right)$$

(48)
Equity Premium

- And the expect return

\[ E_t \frac{Q_{t+1} + D_{t+1}}{Q_t} = \frac{\kappa + 1}{\kappa} = E_t R_{et+1} \]

\[ = \frac{1}{\beta E_t \xi_{t+1}^{1-\gamma}} = \frac{1}{\beta} \exp^{-\frac{1-\gamma}{2} \sigma^2 + \frac{1}{2} (\gamma - 1)^2 \sigma^2} \]

(49)

- so

\[ \log E_t R_{et+1} = - \log(\beta) - \left( -\frac{1-\gamma}{2} \sigma^2 + \frac{1}{2} (\gamma - 1)^2 \sigma^2 \right) \]

(50)

- and the risk premium

\[ \log E_t R_{et+1} - \log R_{ft} \]

\[ = -\left( -\frac{1-\gamma}{2} \sigma^2 + \frac{1}{2} (\gamma - 1)^2 \sigma^2 \right) + \left( \frac{\gamma}{2} \sigma^2 + \frac{1}{2} \gamma^2 \sigma^2 \right) \]

\[ = \gamma \sigma^2 \]

(51)
Equity Premium Puzzle

- In the data, the consumption growth volatility is 0.5508\%, namely $\sigma = 0.5508\%$. The risk premium yield by the model is

  $$(\log E_t R_{et+1} - \log R_{ft}) \times 400 = 400 \gamma \times \sigma^2$$  

- The risk premium in the date is 7.23\% per year.

- We generate the following table

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$E[R_e - R_f]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0121%</td>
</tr>
<tr>
<td>2</td>
<td>0.0243%</td>
</tr>
<tr>
<td>10</td>
<td>0.1214%</td>
</tr>
</tbody>
</table>
Equity Premium Puzzle

- We see that the equity premium is given by
  \[ E[R_e - R_f] = \gamma \sigma^2 \]  
  (53)

- We now check the covariance between consumption growth rate and the risk return:
  \[ \text{cov}(\log(C_{t+1}/D_{t+1}), \log(Q_{t+1} + D_{t+1}/D_t)) = \sigma^2 \]  
  (54)

- so we have
  \[ E[R_e - R_f] = \gamma \text{cov}(g_{t+1}, r_{e,t+1}) \]  
  (55)

- We now show the above results holds for general case.
For the general case, $C_t \neq D_t$

\[
1 = \beta E_t \frac{C_t^\gamma}{C_{t+1}^\gamma} R_{et+1} \\
= \beta E_t \exp(-\gamma g_{t+1} + r_{t+1})
\]

\[
1 = \beta R_{ft} E_t \frac{C_t^\gamma}{C_{t+1}^\gamma} \\
= \beta E_t \exp(-\gamma g_{t+1} + r_{ft})
\]
We do a second order approximation. Define $x_{t+1} - \bar{x}_{t+1} = \hat{x}_{t+1}$, where $\bar{x}_{t+1} = E_t x_{t+1}$.

\[
1 = \beta \exp(-\gamma \hat{g}_{t+1} + \hat{r}_{t+1}) E_t \exp(-\gamma \hat{g}_{t+1} + \hat{r}_{t+1})
\]
\[
= \beta \exp(-\gamma \bar{g}_{t+1} + \bar{r}_{t+1}) \times
\]
\[
E_t \left[ 1 - \gamma \hat{g}_{t+1} + \hat{r}_{t+1} + \frac{1}{2} (\gamma^2 \hat{g}_{t+1}^2 + \hat{r}_{t+1}^2 - 2\gamma \hat{g}_{t+1} \hat{r}_{t+1}) \right]
\]
\[
= \beta \exp(-\gamma \bar{g}_{t+1} + \bar{r}_{t+1}) E_t \left[ 1 + \frac{1}{2} (\gamma^2 \hat{g}_{t+1}^2 + \hat{r}_{t+1}^2 - 2\gamma \hat{g}_{t+1} \hat{r}_{t+1}) \right]
\]

or we have

\[
-\gamma \bar{g}_{t+1} + \bar{r}_{t+1} = -\log(\beta) - \frac{1}{2} E_t (\gamma^2 \hat{g}_{t+1}^2 + \hat{r}_{t+1}^2 - 2\gamma \hat{g}_{t+1} \hat{r}_{t+1})
\]

similary we have

\[
-\gamma \bar{g}_{t+1} + \bar{r}_{ft} = -\log(\beta) - \frac{1}{2} E_t (\gamma^2 \hat{g}_{t+1}^2)
\]
or we have

\[-\gamma \hat{g}_{t+1} + \hat{r}_{t+1} = -\log(\beta) - \frac{1}{2} E_t (\gamma^2 \hat{g}_{t+1}^2 + \hat{r}_{t+1}^2 - 2\gamma \hat{g}_{t+1} \hat{r}_{t+1})\]  \hspace{1cm} (60)

similarly we have

\[-\gamma \hat{g}_{t+1} + \hat{r}_{ft} = -\log(\beta) - \frac{1}{2} E_t (\gamma^2 \hat{g}_{t+1}^2)\]  \hspace{1cm} (61)

the premium is

\[\hat{r}_{t+1} - \hat{r}_{ft} = \gamma E_t \hat{g}_{t+1} \hat{r}_{t+1} - \frac{1}{2} E_t \hat{r}_{t+1}^2\]

\[= \gamma \text{cov}_t(g_{t+1}, r_{t+1}) - \frac{1}{2} \text{var}(r_{t+1})\]  \hspace{1cm} (62)
### Equity Premium Puzzle

<table>
<thead>
<tr>
<th>Country</th>
<th>Sample period</th>
<th>$\bar{a}er_{e}$</th>
<th>$\sigma(\epsilon_{e})$</th>
<th>$\sigma(m)$</th>
<th>$\sigma(\Delta c)$</th>
<th>$\rho(\epsilon_{e}, \Delta c)$</th>
<th>$\text{cov}(\epsilon_{e}, \Delta c)$</th>
<th>RRA(1)</th>
<th>RRA(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>USA</td>
<td>1947.2–1998.3</td>
<td>8.071</td>
<td>15.271</td>
<td>52.853</td>
<td>1.071</td>
<td>0.205</td>
<td>3.354</td>
<td>240.647</td>
<td>49.326</td>
</tr>
<tr>
<td>AUL</td>
<td>1970.1–1998.4</td>
<td>3.885</td>
<td>22.403</td>
<td>17.342</td>
<td>2.059</td>
<td>0.144</td>
<td>6.640</td>
<td>58.511</td>
<td>8.421</td>
</tr>
<tr>
<td>FR</td>
<td>1973.2–1998.3</td>
<td>8.308</td>
<td>23.175</td>
<td>35.848</td>
<td>2.922</td>
<td>-0.093</td>
<td>-6.315</td>
<td>&lt; 0</td>
<td>12.270</td>
</tr>
<tr>
<td>GER</td>
<td>1978.4–1997.3</td>
<td>8.669</td>
<td>20.196</td>
<td>42.922</td>
<td>2.447</td>
<td>0.029</td>
<td>1.446</td>
<td>599.468</td>
<td>17.542</td>
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<tr>
<td>ITA</td>
<td>1971.2–1998.1</td>
<td>4.687</td>
<td>27.068</td>
<td>17.314</td>
<td>1.665</td>
<td>-0.006</td>
<td>-0.252</td>
<td>&lt; 0</td>
<td>10.400</td>
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<tr>
<td>SWD</td>
<td>1970.1–1999.2</td>
<td>11.539</td>
<td>23.518</td>
<td>49.066</td>
<td>1.851</td>
<td>0.015</td>
<td>0.674</td>
<td>1713.197</td>
<td>26.501</td>
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<tr>
<td>SWT</td>
<td>1982.2–1998.4</td>
<td>14.898</td>
<td>21.878</td>
<td>68.098</td>
<td>2.123</td>
<td>-0.112</td>
<td>-5.181</td>
<td>&lt; 0</td>
<td>32.076</td>
</tr>
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<td>USA</td>
<td>1970.1–1998.3</td>
<td>6.353</td>
<td>16.976</td>
<td>37.425</td>
<td>0.909</td>
<td>0.274</td>
<td>4.233</td>
<td>150.100</td>
<td>41.178</td>
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<tr>
<td>SWD</td>
<td>1920–1997</td>
<td>6.540</td>
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<td>6.723</td>
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<td>36.345</td>
<td>6.437</td>
<td>0.495</td>
<td>29.450</td>
<td>22.827</td>
<td>11.293</td>
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</table>
In order to solve the equity premium puzzle, we need incredible high risk aversion coefficient!!! But it create another problem

Risk free rate is equal to

$$\bar{r}_{ft} = - \log(\beta) + \gamma \bar{g}_{t+1} - \frac{\gamma^2}{2} \sigma_g^2$$

High $\gamma$ would leads high risk free rate. One need high $\beta$ to the match the low risk free rate. This is called as risk free rate puzzle.
## Risk free rate puzzle

<table>
<thead>
<tr>
<th>Country</th>
<th>Sample period</th>
<th>( \bar{r}_f )</th>
<th>( \overline{\Delta c} )</th>
<th>( \sigma(\Delta c) )</th>
<th>RRA(1)</th>
<th>TPR(1)</th>
<th>RRA(2)</th>
<th>TPR(2)</th>
</tr>
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<tbody>
<tr>
<td>USA</td>
<td>1947.2–1998.3</td>
<td>0.896</td>
<td>1.951</td>
<td>1.071</td>
<td>240.647</td>
<td>-136.270</td>
<td>49.326</td>
<td>-81.393</td>
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<td>FR</td>
<td>1973.2–1998.3</td>
<td>2.715</td>
<td>1.212</td>
<td>2.922</td>
<td>&lt; 0</td>
<td>N/A</td>
<td>12.270</td>
<td>-5.735</td>
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<tr>
<td>SWT</td>
<td>1982.2–1998.4</td>
<td>1.393</td>
<td>0.559</td>
<td>2.123</td>
<td>&lt; 0</td>
<td>N/A</td>
<td>32.076</td>
<td>6.636</td>
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<tr>
<td>USA</td>
<td>1970.1–1998.3</td>
<td>1.494</td>
<td>1.802</td>
<td>0.909</td>
<td>150.100</td>
<td>-175.916</td>
<td>41.178</td>
<td>-65.701</td>
</tr>
</tbody>
</table>
The utility function is

\[ U = \sum \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \]  \hspace{1cm} (64)

The household problem is to maximize his utility with the constraint:

\[ \sum \beta^t q_t C_t \leq W \]  \hspace{1cm} (65)

where \( W \) is his life-time income.

The intertemporal elasticity of substitution can be obtained from the following equation

\[ \frac{C_t^{-\gamma}}{q_t} = \frac{C_{t+j}^{-\gamma}}{q_{t+j}} \]  \hspace{1cm} (66)
So we have

$$(C_{t+j}/C_t)^{-\gamma} = q_{t+j}/q_t$$  \hfill (67)$$

The relative consumption is inverse function of the relative price. The intertemporal elasticity of substitution is obtained by

$$IES = -\frac{d \ln(C_{t+j}/C_t)}{d \ln(q_{t+j}/q_t)} = \frac{1}{\gamma}$$  \hfill (68)$$
To consider the risk aversion. We look at the one period utility function

\[ U = \frac{C_t^{1-\gamma}}{1 - \gamma} \]  \hspace{1cm} (69)

Consider a certainty consumption \( \mu C_t \) and an uncertainty consumption \( \lambda_t C_t \) which are equivalent to the household. We then must have

\[ E \frac{\lambda_t^{1-\gamma} C_t^{1-\gamma}}{1 - \gamma} = \frac{C_t^{1-\gamma}}{1 - \gamma} \]  \hspace{1cm} (70)

or

\[ E \lambda_t^{1-\gamma} = 1 \]  \hspace{1cm} (71)
We need to solve

$$E\lambda_t^{1-\gamma} = 1$$  \hspace{1cm} (72)

Approximate its around $\bar{\lambda} = E\lambda_t$. We have

$$\lambda_t^{1-\gamma} = \bar{\lambda}^{1-\gamma} + (1 - \gamma)\bar{\lambda}^{-\gamma}[\lambda_t - \bar{\lambda}] - \frac{(1 - \gamma)\gamma}{2}\bar{\lambda}^{-\gamma-1}[\lambda_t - \bar{\lambda}]^2$$  \hspace{1cm} (73)

or we have

$$1 = \bar{\lambda}^{1-\gamma} - \frac{(1 - \gamma)\gamma}{2}\bar{\lambda}^{1-\gamma}E_t\left[\frac{\lambda_t - \bar{\lambda}}{\bar{\lambda}}\right]^2$$  \hspace{1cm} (74)
The separable utility

\[ \bar{\lambda} = \left[ \frac{1}{1 - \frac{(1-\gamma)\lambda}{2}\sigma^2} \right]^{\frac{1}{1-\gamma}} \]  \hspace{1cm} (75)

or

\[ \bar{\lambda} = 1 + \frac{\gamma}{2}\sigma^2 \]  \hspace{1cm} (76)

so

\[ \bar{\lambda} - 1 = \frac{\gamma}{2}\sigma^2 \]  \hspace{1cm} (77)

the premium is

a high risk aversion coefficient \( \gamma \) require a high premium.
Now consider the general utility function

\[ U = \sum \beta^t u(C_t) \]  (78)

Similarly we have

\[ \frac{u'(C_t)}{p_t} = \frac{u'(C_{t+j})}{p_{t+j}} \]  (79)

This implies

\[ \frac{u'(C_{t+j})}{u'(C_t)} = \frac{p_{t+j}}{p_t} \]  (80)

Consider small price change \( p_{t+j} = p_t + \Delta p_t \)
The separable utility

- We approximate $u'(C_{t+j})$, we have

$$u'(C_{t+j}) = u'(C_t) + u''(C_t) C_t \left( \frac{C_{t+j} - C_t}{C_t} \right)$$  \hspace{1cm} (81)

$$\frac{u'(C_{t+j})}{u'(C_t)} = \frac{p_{t+j}}{p_t}$$  \hspace{1cm} (82)

- so we have

$$1 + \frac{u''(C_t) C_t}{u'(C_t)} \left( \frac{C_{t+j} - C_t}{C_t} \right) = 1 + \frac{\Delta p_t}{p_t}$$  \hspace{1cm} (83)

- or we have

$$\left( \frac{\Delta C_t}{C_t} \cdot \frac{\Delta p_t}{p_t} \right) = \frac{1}{\frac{u''(C_t) C_t}{u'(C_t)}} = IES$$
The separable utility

To derive the risk aversion coefficient, we define

\[ E_t u(\lambda_t C_t) = u(C_t) \]  \hspace{1cm} (84)

Again approximate \( u(\lambda_t C_t) \) around \( U(C_t) \), we have

\[ u(\lambda_t C_t) = u(C_t) + u'(C_t)[\lambda_t - 1]C_t + \frac{u''(C_t)}{2}[\lambda_t - 1]^2C_t^2 \]  \hspace{1cm} (85)

or we have

\[ E\lambda_t - 1 = -\frac{1}{2} \frac{u''(C_t)C_t}{u'(C_t)} E_t[\lambda_t - 1]^2 \]  \hspace{1cm} (86)

Notice \( \text{var}(\lambda_t - 1) = E[\lambda_t - 1]^2 - (E\lambda_t - 1)^2 \), the above equation suggest

\[ \text{var}(\lambda_t - 1) = \sigma^2 = E[\lambda_t - 1]^2 + o(\sigma^3) \]  \hspace{1cm} (87)

So the premium depends on \( \frac{u''(C_t)C_t}{u'(C_t)} \)
Generally speaking, the risk aversion coefficient and the intertemporal substitution coefficients in the time separable utility case are captured by the same parameters.

However intertemporal substitution and risk aversion are two different concepts. So it is not clear that these should be the same.
Utility function

\[ U_t = \left[ c_t^\rho + \beta (E_t U_{t+1}^\alpha)^{\rho \over \alpha} \right]^{1 \over \rho} \]  
(88)

If \( \rho = \alpha \). The utility function becomes

\[ U_t^\rho = c_t^\rho + \beta (E_t U_{t+1}^\rho) \]  
(89)

If we iterate forward, we have

\[ U_t^\rho = E_t \sum \beta^j c_{t+j}^\rho \]  
(90)

which is the usual utility function.

If there is no uncertainty, again we have

\[ U_t^\rho = \sum \beta^j c_{t+j}^\rho \]  
(91)

so \( \rho \) captures the intertemporal substitution. And \( \alpha \) captures the risk attitudes.
Household problem

- Utility function

\[ U_t = \left[ c_t^\rho + \beta (E_t U_{t+1}^\alpha)^{\rho \alpha} \right]^{\frac{1}{\rho}} \quad (92) \]

- The budget constraint:

\[ c_t + \sum_{i=1}^{N} S_{i,t+1} = \sum_{i=1}^{N} S_{t,i} R_{i,t} \quad (93) \]

- Define the total asset as

\[ A_t = \sum_{i=1}^{N} S_{t,i} R_{i,t} \quad (94) \]
Household problem

- The net asset in the next period is then

\[ A_{t+1} = \sum_{i=1}^{N} S_{t+1,i} R_{t+1,i} \]

\[ = \sum_{i=1}^{N} S_{i,t+1} \left[ \sum_{i=1}^{N} \omega_{t,i} R_{t,i} \right] \]

\[ = (A_t - c_t) \omega_t' R_t \]

\[ = (A_t - c_t) M_t \]

- Where \( M_t \) is the total portfolio return. Take \( M_t \) as given, we have:

\[ A_{t+1} = (A_t - c_t) M_{t+1} \quad (95) \]
Household problem

- We can solve the above problem by guess and verification strategy. Guess the value function is linear in wealth. Namely

\[ U_t = \phi_t A_t \]  \hspace{1cm} (96)

- where \( \phi_t \) remains to be determine. With the above guess, we have

\[
\max U_t = \left\{ c_t^\rho + \beta (A_t - c_t)^\rho \left[ E_t \phi_{t+1}^\alpha m_{t+1}^\alpha \right]^{\frac{\rho}{\alpha}} \right\}^{\frac{1}{\rho}}
\]  \hspace{1cm} (97)

- Define \( \mu^* = \left[ E_t \phi_{t+1}^\alpha m_{t+1}^\alpha \right]^{\frac{1}{\alpha}} \) We then have the foc with respect to consumption is
Household problem

- Define $\mu^* = \left[ E_t \theta_{t+1} M_{t+1} \right]^{\frac{1}{\alpha}}$. We then have the foc with respect to consumption is

$$\max_{c_t} U_t = \left\{ c_t^\rho + \beta (A_t - c_t)^\rho \mu^* \right\}^{\frac{1}{\rho}}$$

(98)

- or it yields

$$c_t^{\rho - 1} = \beta (A_t - c_t)^{\rho - 1} \mu^*$$

(99)

- we can then solve $\mu^* \mu^\rho$
Household problem

- We have obtained

\[ c_t^{\rho-1} = \beta (A_t - c_t)^{\rho-1} \mu^{*\rho} \]  \hspace{1cm} (100)

- so we have

\[ U_t = \left\{ c_t^\rho + \beta (A_t - c_t)^\rho \frac{c_t^{\rho-1}}{\beta (A_t - c_t)^{\rho-1}} \right\}^{\frac{1}{\rho}} \]

\[ = \left\{ c_t^\rho + (A_t - c_t) c_t^{\rho-1} \right\}^{\frac{1}{\rho}} \]

\[ = \phi_t A_t \]  \hspace{1cm} (101)

- or

\[ \phi_t = \left( \frac{c_t}{A_t} \right)^{\frac{\rho-1}{\rho}} \]  \hspace{1cm} (102)
By definition we have

\[ \mu^{\ast \rho} = \left[ E \left( \frac{c_{t+1}}{A_{t+1}} \right)^{\frac{\rho-1}{\rho} \alpha} m_{t+1}^{\alpha} \right]^{\frac{\rho}{\alpha}} \]  

(103)

\[ = (A_t - c_t)^{1-\rho} \left[ E \left( \frac{c_{t+1}}{m_{t+1}} \right)^{\frac{\rho-1}{\rho} \alpha} m_{t+1}^{\alpha} \right]^{\frac{\rho}{\alpha}} \]

And by

\[ c_t^{\rho-1} = \beta (A_t - c_t)^{\rho-1} \mu^{\ast \rho} \]  

(104)

We obtain

\[ c_t^{(\rho-1) \frac{\alpha}{\rho}} = \beta^{\frac{\alpha}{\rho}} \left[ E \left( \frac{c_{t+1}}{m_{t+1}} \right)^{\frac{\rho-1}{\rho} \alpha} m_{t+1}^{\alpha} \right] \]  

(105)
Household problem

- Let $\theta = \frac{\alpha}{\rho}$, we then have

$$E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho-1} m_{t+1} \right]^\theta = 1 \quad (106)$$

- So far we have obtain the Euler equation for consumption. Now we try to determine the portfolio. Given $c_t^*$, the problem is to solve

$$\max_{\omega_{i,t+1}} \left\{ E_t \phi_{t+1}^\alpha \left( \sum_{i=1}^N \omega_{i,t+1} R_{i,t+1} \right)^\alpha \right\}^{\frac{1}{\alpha}} \quad (107)$$

- with the constraint

$$\sum_{i=1}^N \omega_{i,t+1} = 1 \quad (108)$$
Household problem

- Again this is a static problem: The first order conditions are

\[
\left\{ E_t \phi_{t+1}^\alpha \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^\alpha \right\}^{\frac{1}{\alpha}-1} \times \\
E_t \phi_{t+1}^\alpha \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^{\alpha-1} R_{i,t+1} = \lambda_t
\]

(109)

- Multiply this equation by \( \omega_{i,t+1} \), sum over \( i \) to derive:

\[
\left\{ E_t \phi_{t+1}^\alpha \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^\alpha \right\}^{\frac{1}{\alpha}} = \lambda_t
\]

(110)
Household problem

- This yields

\[
E_t \phi_{t+1}^\alpha \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^{\alpha-1} R_{i,t+1}
\]

\[
= E_t \phi_{t+1}^\alpha \left( \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1} \right)^{\alpha}
\]  

(111)

- by definition

\[
m_{t+1} = \sum_{i=1}^{N} \omega_{i,t+1} R_{i,t+1}
\]  

(112)
We now substitute $\phi_{t+1}$ out by

$$\phi_t = \left( \frac{c_t}{A_t} \right)^{\frac{\rho-1}{\rho}} \tag{113}$$

or we have

$$E_t \left( \frac{c_{t+1}}{A_{t+1}} \right)^{\theta(\rho-1)} m_{t+1}^{\alpha-1} R_{i,t+1} = E_t \left( \frac{c_{t+1}}{A_{t+1}} \right)^{\theta(\rho-1)} m_{t+1}^\alpha \tag{114}$$

finally we have

$$A_{t+1} = (A_t - c_t) m_{t+1} \tag{115}$$
Household problem

- Further simplification yields

\[ E_t \left( \frac{c_{t+1}}{m_{t+1}} \right)^{\theta(\rho-1)} m_{t+1}^{\theta-1} R_{i,t+1} = E_t \left( \frac{c_{t+1}}{m_{t+1}} \right)^{\theta(\rho-1)} m_{t+1}^\theta \]

\[ = E_t c_{t+1}^{\theta(\rho-1)} m_{t+1}^\theta \quad (116) \]

- Finally use that

\[ E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho-1} m_{t+1} \right]^\theta = 1 \quad (117) \]

- Multiplying both side by \( c_t^{-\theta(\rho-1)} \beta^\theta \) we yields

\[ E_t \beta^\theta \left( \frac{c_{t+1}}{c_t} \right)^{\theta(\rho-1)} m_{t+1}^{\theta-1} R_{i,t+1} = 1 \quad (118) \]
Consider two assets. One is the stock, the other is risk-free bonds. The bond has zero supply. So in equilibrium we have

$$\omega_{b,t+1} = 0$$

(119)

and

$$\omega_{et+1} = 1$$

(120)

where $\omega_{b,t+1}$ is the weight on bond and $\omega_{et+1}$ is the weight on stock.

Hence we have

$$m_{t+1} = R_{et+1}$$

(121)
We hence have

\[ E_t \beta^\theta \left( \frac{c_{t+1}}{c_t} \right)^{\theta(\rho-1)} R_{et+1}^\theta = 1 \]  

(122)

and

\[ E_t \beta^\theta \left( \frac{c_{t+1}}{c_t} \right)^{\theta(\rho-1)} R_{et+1}^{\theta-1} R_{ft} = 1 \]  

(123)

We can rewrite them in exponential terms

\[ \beta^\theta E_t e^{\theta(\rho-1)\Delta c_{t+1} + (\theta-1) \log m_{t+1} + r_{i,t+1}} = 1 \]  

(124)
Assume normal distribution for $\Delta c_{t+1}, \log m_{t+1}, r_{it+1}$, we hence has

$$\theta \log \beta + \theta (\rho - 1) E \Delta c_{t+1} + (\theta - 1) Em_{t+1} + Er_{i,t+1}$$

$$+ \frac{1}{2} \sigma^2 = 0$$

where

$$\sigma^2 = \theta^2 (\rho - 1)^2 \sigma^2_c + (\theta - 1)^2 \sigma^2_m + \sigma^2_i + 2\theta (\rho - 1) \sigma_{c,i} + 2\theta (\rho - 1) (\theta - 1) \sigma_{c,m} + 2(\theta - 1) \sigma_{i,m}$$

where $\sigma_{c,i}$ is the covariance between consumption growth and the log return.

For risk free return, we have $\sigma^2_f = 0; \sigma_{c,f} = 0; \sigma_{i,m} = 0$. 
Hence the risk premium is

\[ Er_{t+1} - r_{f,t+1} \approx \theta (1 - \rho) \text{cov}(\Delta c_{t+1}, r_{et+1}) + (1 - \theta)\sigma_e^2 - \frac{1}{2}\sigma_e^2 \]  

(127)

in the case \( \rho = \alpha \), so \( \theta = 1 \), this reduces to

\[ Er_{t+1} - r_{f,t+1} \approx (1 - \rho) \text{cov}(\Delta c_{t+1}, r_{et+1}) - \frac{1}{2}\sigma_e^2 \]  

(128)

which is the same as before.

We requires a very large \( \gamma \)

\[ \gamma = 1 - \rho \]  

(129)

and in this case, we can explain the risk premium puzzle but generate risk free rate puzzle.
Risk Premium

- The E-Z preference can solve both puzzle. To see this. Let $\rho = 1$. And in such case, the risk free rate is given by

$$Er_{t+1} - r_{f,t+1} \simeq \theta(1 - \rho) \operatorname{cov}(\Delta c_{t+1}, r_{et+1}) + (1 - \theta)\sigma_e^2 - \frac{1}{2}\sigma_e^2$$

(130)

$$r_{ft} = -\theta \log \beta + (1 - \theta)E_t r_{et+1} - \frac{1}{2}[(\theta - 1)^2\sigma_e^2]$$

(131)

- Combine the above two equations we have

$$\theta r_{ft} + (\theta - 1)[(1 - \theta)\sigma_e^2 - \frac{1}{2}\sigma_e^2] = -\theta \log \beta - \frac{1}{2}[(\theta - 1)^2\sigma_e^2]$$

(132)

$$\theta r_{ft} = -\theta \log \beta - \frac{1}{2}[(\theta - 1)^2 + 2(\theta - 1)(1 - \theta) - (\theta - 1)]\sigma_e^2$$

(133)

$$= -\theta \log \beta + \frac{1}{2}[(\theta - 1)\theta]\sigma_e^2$$

- or we have

$$r_{ft} = -\log \beta + \frac{1}{2}(\theta - 1)\sigma_e^2 < -\log \beta$$

(134)
The representative household solves:

$$\max \ U_t = \left\{ \left[ c_t^\theta (1 - n_t)^{1-\theta} \right]^\rho + \beta \left( E_t U_{t+1}^\gamma \right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho}}$$

s. t

$$c_t + k_{t+1} = A_t k_t^\alpha n_t^{1-\alpha} + (1 - \delta) k_t$$
Set-up the Lagrangian function

\[ U(k_t, A_t) = \left\{ \left[ c_t^\theta (1 - n_t)^{1-\theta} \right]^\rho + \beta (E_t U_{t+1}^\gamma)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho}} + \lambda_t [A_t k_t^\alpha n_t^{1-\alpha} + (1 - \delta) k_t - c_t - k_{t+1}] \] (135)

The first order conditions with respect to \(c_t, n_t\) and \(k_{t+1}\) are:

\[ \left\{ \left[ c_t^\theta (1 - n_t)^{1-\theta} \right]^\rho + \beta (E_t U_{t+1}^\gamma)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho}-1} \times \left[ c_t^\theta (1 - n_t)^{1-\theta} \right]^{\rho-1} \theta c_t^{\theta-1} = \lambda_t \] (136)
With respect to labor is

\[
\begin{align*}
\left\{ \left[ c_t^\theta (1 - n_t)^{1 - \theta} \right]^\rho + \beta \left( E_t U_{t+1}^\gamma \right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho - 1}} \\
\times \left[ c_t^\theta (1 - n_t)^{1 - \theta} \right]^{\rho - 1} (1 - \theta) n_t^{\theta - 1} \\
= \lambda_t \frac{(1 - \alpha) y_t}{n_t}
\end{align*}
\]

(137)

and with respect to capital is

\[
\begin{align*}
\left\{ \left[ c_t^\theta (1 - n_t)^{1 - \theta} \right]^\rho + \beta \left( E_t U_{t+1}^\gamma \right)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho - 1}} \\
\times \beta \left( E_t U_{t+1}^\gamma \right)^{\frac{\rho}{\gamma} - 1} \\
\times EU_{t+1}^{\gamma - 1} \frac{\partial U'}{\partial k'} \\
= \lambda_t
\end{align*}
\]
Envelop theory

\[
\frac{\partial U(k_t, A_t)}{\partial k_t} = \lambda_t \left[ \alpha \frac{y_t}{k_t} + (1 - \delta) \right]
\]  \hspace{1cm} (138)

So the Euler equation is

\[
U_t^{1-\rho} \left[ c_t^{\theta} (1 - n_t)^{1-\theta} \right]^{\rho-1} \theta c_t^{\theta-1} = U_t^{1-\rho} \beta \left( E_t U_t^{\gamma} \right)^{\rho-1} \gamma
\]

\[
\times E_t U_t^{\gamma-\rho} \left[ c_{t+1}^{\theta} (1 - n_{t+1})^{1-\theta} \right]^{\rho-1} \theta c_{t+1}^{\theta-1}
\]  \hspace{1cm} (139)

it can be reduced further to

\[
\left[ c_t^{\theta} (1 - n_t)^{1-\theta} \right]^{\rho-1} c_t^{\theta-1} = \beta \left( E_t U_t^{\gamma} \right)^{\rho-\gamma} \gamma
\]

\[
\times E_t U_t^{\gamma-\rho} \left[ c_{t+1}^{\theta} (1 - n_{t+1})^{1-\theta} \right]^{\rho-1} \theta c_{t+1}^{\theta-1}
\]  \hspace{1cm} (140)
in summary we have

\[
\left[ c_t^\theta (1 - n_t)^{1-\theta} \right]^{\rho-1} c_t^{\theta-1} = \beta (E_t U_{t+1}^\gamma)^{\frac{\rho-\gamma}{\gamma}} \times E_t U_{t+1}^{\gamma-\rho} \left[ c_{t+1}^\theta (1 - n_{t+1})^{1-\theta} \right]^{\rho-1} c_{t+1}^{\theta-1} \tag{141}
\]

\[
\theta c_t^{\theta-1} \frac{(1 - \alpha) y_t}{n_t} = (1 - \theta) n_t^{-\theta} \tag{n}
\]

\[
c_t + k_{t+1} = A_t k_t^\alpha n_t^{1-\alpha} + (1 - \delta) k_t \tag{c}
\]

\[
U_t = \left\{ \left[ c_t^\theta (1 - n_t)^{1-\theta} \right]^{\rho} + \beta (E_t U_{t+1}^\gamma)^{\frac{\rho}{\gamma}} \right\}^{\frac{1}{\rho}} \tag{U}
\]

\[
y_t = A_t k_t^\alpha n_t^{1-\alpha} \tag{y}
\]