Notes on the Real Business Cycle Model

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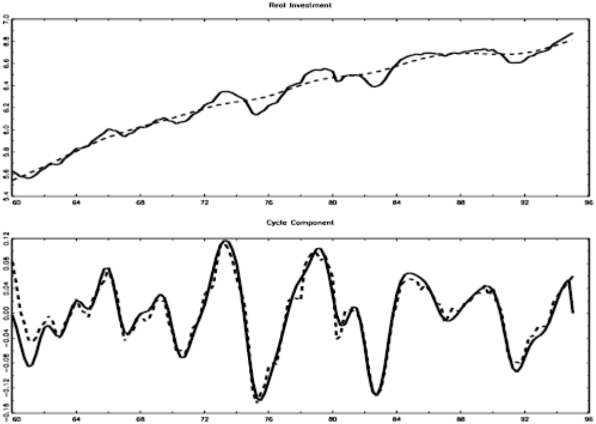
Introduction: Basic Facts about Business Cycle



Introduction: Basic Facts about Business Cycle 2

Table 1: Sample Moments: U.S. 1948:1 - 1994:2

var	*std.*	*rel. std*	$cor(x_t, y_t)$	$cor(x_t, x_{t-1})$
y	0.020	1.0	1.0	0.95
c	0.008	0.4	0.8	0.96
i	0.050	2.6	0.9	0.96
n	0.015	0.8	0.8	0.96
y/n	0.010	0.5	0.6	0.93

Introduction: Basic Facts about Business Cycle 2

Table 2: Behavior of components of output in recessions

Components	ave. share in GDP	ave. share in fall in GDP
Consumption		
– durables	8%	16%
– nondurables	26%	11%
– services	30%	9%
Investment		
– residential	5%	21%
– fixed business	11%	12%
– inventories	0.7%	41%
Net Exports	−0.4%	−12%
Government Purchases	21%	3%

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- How Should Government Policy Respond to Business Cycles?

Two Major Schools of Thoughts:

The Classical School–>Doctrine: Supply determines demand

- supply shocks are the major source of business cycles.

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- Features: Very rigorous but against common sense.

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- Propagation mechanisms include interest rates, portfolio allocations, asset bubbles (including the stock market), banking and financial crises, and international trade and currency crises.
- Policy recommendation: Intervene
- Features: Appealing to common sense but very vague and imprecise.

Key Assumptions:

- prices adjust instantaneously to clear markets
- rational expectations
- perfect competition
- perfect risk sharing
- no asymmetric information
- no externalities

A Benchmark Model:

A social planner (or representative agent) chooses paths of consumption, investment and employment (hours) to solve

- objective function

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \gamma \log \left(1 - n_t\right)\right] \quad (1)$$

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- Resource Constraint

$$c_t + k_{t+1} - (1 - \delta) k_t = A_t k_t^{\alpha} n_t^{1-\alpha}, \qquad (2)$$

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- Shock Process

$$\log A_t = \rho \log A_{t-1} + \varepsilon_t. \qquad (3)$$

The Lagrangian:

- The Lagrangian is given by

$$L = E_0 \left\{ \begin{array}{c} \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \gamma \log \left(1 - n_t \right) \right] \\ + \lambda_t \left[A_t k_t^{\alpha} n_t^{1-\alpha} - c_t - k_{t+1} + \left(1 - \delta \right) k_t \right] \end{array} \right\}. \quad (4)$$

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$$k_t ; A_t \tag{5}$$

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- choice variables in period t

$$c_t, n_t, \lambda_t, k_{t+1} \tag{6}$$

The First Order Conditions:

- with respect to consumption :

$$\frac{1}{c_t} = \lambda_t \tag{7}$$

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- with respect to k_{t+1}

$$\lambda_t = \beta E_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha - 1} n_{t+1}^{1-\alpha} + 1 - \delta \right) \right] \tag{9}$$

The First Order Conditions (continued):

- with respect to λ_t :

$$c_t + k_{t+1} - (1-\delta)\, k_t = A_t k_t^{\alpha} n_t^{1-\alpha}, \tag{10}$$

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$$\lim_{T\to\infty} E_0 \beta^T \lambda_T k_{T+1} = 0, \tag{11}$$

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- and the law of motion for technology

$$\log A_t = \rho \log A_{t-1} + \varepsilon_t. \tag{12}$$

Equilibrium:

An equilibrium is a set of decision rules:

$$x_t = x\left(k_t, A_t\right)$$

for $x = \{c_t, k_{t+1}, n_t, \lambda_t\}$ such that equations (7)-(12) are satisfied.

Steady State :

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- and (10) implies

$$\frac{c}{y} = 1 - \delta \frac{k}{y}. \tag{15}$$

Steady State (continued) :

- Hence, the great ratios are given by

$$\frac{i}{y} = \frac{\delta\alpha\beta}{1-\beta(1-\delta)} \tag{16}$$

$$\frac{c}{y} = 1 - \frac{\delta\alpha\beta}{1-\beta(1-\delta)} \tag{17}$$

$$\frac{k}{y} = \frac{\alpha\beta}{1-\beta(1-\delta)}. \tag{18}$$

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- Note that the steady-state rate of saving is given by

$$s^* = \frac{i}{y} = \frac{\delta\alpha\beta}{1-\beta(1-\delta)}. \tag{19}$$

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- and by $\frac{y}{k} = A\left(\frac{n}{k}\right)^{1-\alpha}$, we have

$$k^{*} = \left[A\left(\frac{k}{y}\right)\right]^{\frac{1}{1-\alpha}} n^{*} \tag{22}$$

$$y^{*} = A\left(k^{*}\right)^{\alpha}\left(n^{*}\right)^{1-\alpha};\, i^{*} = s^{*}y^{*};\, c^{*} = \left(1-s^{*}\right)y^{*} \tag{23}$$

A Decentralized Version-The firms :

- Aggregate Production Technology:

$$Y_t = A_t K_t^{\alpha} (n_t L_t)^{1-\alpha}, \tag{24}$$

where n is hours per worker and L is the labor force (without loss of generality, assuming its growth rate be zero).

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- which implies the following factor demand functions:

$$r_t + \delta = \alpha A_t k_t^{\alpha-1} n_t^{1-\alpha} \tag{27}$$

$$w_t = \left(1-\alpha\right) A_t k_t^{\alpha} n_t^{-\alpha}. \tag{28}$$

A Decentralized Version-The households :

- A representative worker's problem:

$$\max_{\{c_t, s_{t+1}, n_t^s\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \gamma \log\left(1 - n_t^s\right)\right] \qquad (29)$$

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- First order conditions:

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ \frac{\gamma}{1 - n_t^s} &= \lambda_t w_t \\ \lambda_t &= \beta E_t \lambda_{t+1} \left(1 + r_{t+1}\right). \end{aligned} \tag{31}$$

A Decentralized Version-Equilibrium :

- Equilibrium: In equilibrium, prices clear the markets and supply meets demand:

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- Hence we have:

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$$\lambda_t = \beta E_t \lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta \right) \tag{36}$$

$$c_t + k_{t+1} = (1-\delta) k_t + A_t k_t^{\alpha} n_t^{1-\alpha}, \tag{37}$$

A special case :

- Equations (3) and (4) become

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \alpha \frac{y_{t+1}}{k_{t+1}} \right] \tag{38}$$

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- Then we have,

$$\frac{1}{y_t} = \beta E_t \left[\frac{1}{y_{t+1}} \alpha \frac{y_{t+1}}{k_{t+1}} \right] = \beta \alpha E_t \frac{1}{k_{t+1}}, \tag{41}$$

which implies

$$k_{t+1} = \beta \alpha y_t. \tag{42}$$

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- Substituting this into (39) gives

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- (8) then implies

$$\begin{aligned} \frac{\gamma}{1-n_t} &= \frac{1}{(1-\beta\alpha)\, y_t}\left[(1-\alpha)\frac{y_t}{n_t}\right] \\ &= \frac{1-\alpha}{1-\beta\alpha}\frac{1}{n_t}, \end{aligned} \tag{44}$$

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- which implies

$$n_t = \frac{\frac{1-\alpha}{1-\beta\alpha}}{\gamma + \frac{1-\alpha}{1-\beta\alpha}} \in (0, 1). \tag{45}$$

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- we have already obtained

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- Hence the decision rules are given by

$$y_t = A_t k_t^\alpha \bar{n}^{1-\alpha} \tag{47}$$

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$$k_{t+1} = \beta\alpha A_t k_t^\alpha \bar{n}^{1-\alpha}. \tag{49}$$

A special case -Log linearization:

- take log we have

$$\hat{y}_t = \hat{A}_t + \alpha \hat{k}_t \quad (50)$$

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- Since

$$\begin{aligned} \hat{A}_t &= \rho \hat{A}_{t-1} + \varepsilon_t \\ &= \frac{1}{1-\rho L}\varepsilon_t = \sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j}, \end{aligned} \quad (53)$$

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- the decision rule for capital can be rewritten as a moving-average process:

$$\hat{k}_{t+1} = \frac{1}{1-\alpha L}\hat{A}_t = \frac{1}{(1-\alpha L)(1-\rho L)}\varepsilon_t,$$

A special case -Log linearization(continued):

- or as an AR(2) process:

$$\left(1-\alpha L\right)\left(1-\rho L\right)\hat{k}_{t+1} = \varepsilon_t \quad (54)$$

$$\left(1-\left(\alpha+\rho\right)L+\alpha\rho L^2\right)\hat{k}_{t+1} = \varepsilon_t \quad (55)$$

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- or

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- Utilizing (50), we can also express consumption and output as ARMA (p, q) processes $(p = 2, q = 0)$:

$$x_t = \frac{1}{1-\rho L}\varepsilon_t + \frac{\alpha}{\left(1-\alpha L\right)\left(1-\rho L\right)}\varepsilon_{t-1} \tag{57}$$

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- or as an AR(2) process:

$$\left(1-\alpha L\right)\left(1-\rho L\right)\hat{k}_{t+1} = \varepsilon_t \tag{54}$$

$$\left(1-\left(\alpha+\rho\right)L+\alpha\rho L^2\right)\hat{k}_{t+1} = \varepsilon_t \tag{55}$$

- or

$$\hat{k}_{t+1} = \left(\alpha+\rho\right)\hat{k}_t - \alpha\rho\hat{k}_{t-1} + \varepsilon_t. \tag{56}$$

- Utilizing (50), we can also express consumption and output as ARMA (p, q) processes $(p = 2, q = 0)$:

$$x_t = \frac{1}{1-\rho L}\varepsilon_t + \frac{\alpha}{\left(1-\alpha L\right)\left(1-\rho L\right)}\varepsilon_{t-1} \tag{57}$$

- or

$$\begin{aligned}\left(1-\alpha L\right)\left(1-\rho L\right)x_t &= \left(1-\alpha L\right)\varepsilon_t + \alpha\varepsilon_{t-1} \\ &= \varepsilon_t.\end{aligned} \tag{58}$$

A special case -impulse responses:

- Consider the impulse responses of capital to a one unit increase in technology at time t. Since

$$\hat{k}_{t+1} = (\alpha + \rho)\,\hat{k}_t - \alpha\rho\hat{k}_{t-1} + \varepsilon_t, \tag{59}$$

A special case -impulse responses:

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$$\hat{k}_{t+1} = (\alpha + \rho)\,\hat{k}_t - \alpha\rho\hat{k}_{t-1} + \varepsilon_t, \tag{59}$$

- or using the state-space representation, we have

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{k}_t \end{pmatrix} = \begin{pmatrix} \alpha+\rho & -\alpha\rho \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{k}_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{pmatrix}, \tag{60}$$

Log linearization:

- First-order conditions:

$$\frac{1}{c_t} = \lambda_t \tag{61}$$

$$\frac{\gamma}{1 - n_t} = \lambda_t \left[(1 - \alpha) A_t k_t^{\alpha} n_t^{-\alpha} \right] \tag{62}$$

$$\lambda_t = \beta E_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta \right) \right] \tag{63}$$

$$c_t + k_{t+1} - (1 - \delta) k_t = A_t k_t^{\alpha} n_t^{1-\alpha}. \tag{64}$$

Log linearization (continued):

- Denote $\hat{x}_t = \frac{x_t - x}{x} \simeq \log(x_t) - \log(x)$ as the percentage change from steady-state.

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yields

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- log-linearizing equation

$$y_t = x_{1t}^{\alpha} x_{2t}^{\beta} \tag{67}$$

yieds

$$\hat{y}_t = \alpha \hat{x}_{1t} + \beta \hat{x}_{2t} \tag{68}$$

Log linearization (continued):

- log-linearizing equation

$$y_t = x_{1t} + x_{2t} \tag{69}$$

yields

$$\hat{y}_t = \frac{x_1}{x_1 + x_2}\hat{x}_{1t} + \frac{x_2}{x_1 + x_2}\hat{x}_{2t} \tag{70}$$

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- log-linearizing equation

$$y_t = \sum_{j=1}^{J} x_{jt} \tag{71}$$

yieds

$$\hat{y}_t = \sum_{j=1}^{J} \left(\frac{x_j}{y}\right)\hat{x}_{jt} \tag{72}$$

Log linearization (continued):

- log-linearizing equation

$$y_t = E_t[x_{1t+1} + x_{2t+1}] \tag{73}$$

yields

$$\hat{y}_t = \frac{x_1}{x_1 + x_2} E_t \hat{x}_{1t+1} + \frac{x_2}{x_1 + x_2} E_t \hat{x}_{2t+1} \tag{74}$$

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yieds

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Log linearizing the f.o.cs:

- Log linearization the equation

$$\frac{1}{c_t} = \lambda_t \tag{77}$$

gives

$$-\hat{c}_t = \hat{\lambda}_t \tag{78}$$

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yieds

$$\frac{n}{1-n}\hat{n}_t = \hat{\lambda}_t + \hat{A}_t + \alpha \hat{k}_t - \alpha \hat{n}_t \tag{80}$$

where $s_c = \frac{c}{y}, s_i = \frac{i}{y}$.

Log linearizing the f.o.cs(continued):

- Log linearization the equation

$$\lambda_t = \beta E_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta \right) \right] \tag{81}$$

yields

$$\hat{\lambda}_t = E_t \left\{ \hat{\lambda}_{t+1} + (1 - \beta (1 - \delta)) \left[(\alpha - 1) \hat{k}_{t+1} + (1 - \alpha) \hat{n}_{t+1} + \hat{A}_t \right] \right\} \tag{82}$$

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- and log-linearization the equation

$$c_t + k_{t+1} - (1 - \delta) k_t = A_t k_t^{\alpha} n_t^{1-\alpha}. \tag{83}$$

yields

$$s_c \hat{c}_t + s_i \left[\frac{1}{\delta}\hat{k}_{t+1} - \frac{(1-\delta)}{\delta}\hat{k}_t\right] = \hat{A}_t + \alpha \hat{k}_t + (1 - \alpha) \hat{n}_t \tag{84}$$

Log linearizing the f.o.cs(continued):

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$$\lambda_t = \beta E_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta \right) \right] \tag{81}$$

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$$c_t + k_{t+1} - (1 - \delta) k_t = A_t k_t^{\alpha} n_t^{1-\alpha}. \tag{83}$$

yields

$$s_c \hat{c}_t + s_i \left[\frac{1}{\delta} \hat{k}_{t+1} - \frac{(1-\delta)}{\delta} \hat{k}_t \right] = \hat{A}_t + \alpha \hat{k}_t + (1 - \alpha) \hat{n}_t \tag{84}$$

- and the technology follows

$$\hat{A}_t = \rho \hat{A}_{t-1} + \varepsilon_t \tag{85}$$

Reduced forms:

- we can write consumption and labor as

$$\begin{pmatrix} \hat{c}_t \\ \hat{n}_t \end{pmatrix} = A_1 \begin{pmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{pmatrix} + A_2 \hat{A}_t \tag{86}$$

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- and equation (82) and (84) become

$$E_t \begin{pmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{pmatrix} = B \begin{pmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{pmatrix} + R_1 E_t \hat{A}_{t+1} + R_2 \hat{A}_t . \tag{87}$$

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- A unique equilibrium exists if one of the eigenvalues of B lies outside the unit circle and the other lies inside the unit circle.

Solution Method 1:

- (Remember to put back E_t operator later)

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{pmatrix} = P \Lambda P^{-1} \begin{pmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{pmatrix} + R_1 \hat{A}_{t+1} + R_2 \hat{A}_t, \tag{88}$$

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- or

$$\begin{pmatrix} x_{1t+1} \\ x_{2t+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} + \tilde{R}_1 \hat{A}_{t+1} + \tilde{R}_2 \hat{A}_t. \tag{90}$$

Solution Method 1 (continued):

- Suppose $|\lambda_2| > 1$, then

$$E_t x_{2t+1} = \lambda_2 x_t + r_1 E_t \hat{A}_{t+1} + r_2 \hat{A}_t, \tag{91}$$

- or

$$x_{2t} = \frac{1}{\lambda_2} E_t x_{2t+1} - \frac{1}{\lambda_2} \left[r_1 E_t \hat{A}_{t+1} + r_2 \hat{A}_t \right].$$

Solution Method 1 (continued):

- Denote the foward operator as F,

$$Fx_t = E_t x_{t+1}, FFx_t = FE_t x_{t+1} = E_t[E_{t+1} x_{t+2}] = E_t x_{t+2} \tag{92}$$

Solution Method 1 (continued):

- Denote the foward operator as F,

$$Fx_t = E_t x_{t+1}, FFx_t = FE_t x_{t+1} = E_t[E_{t+1}x_{t+2}] = E_t x_{t+2} \tag{92}$$

- Iterating this forward and applying the law of iterated expectations, a stationary solution for this difference equation is given by

$$\begin{aligned}
x_{2t} &= \frac{-\frac{1}{\lambda_2}}{1-\frac{1}{\lambda_2}F}\left\{r_1 E_t \hat{A}_{t+1} + r_2 \hat{A}_t\right\} \\
&= -\frac{r_1}{\lambda_2}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda_2}\right)^j E_t\hat{A}_{t+1+j} - \frac{r_2}{\lambda_2}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda_2}\right)^j E_t\hat{A}_{t+j} \\
&= -\frac{r_1}{\lambda_2}\frac{\rho}{1-\frac{\rho}{\lambda_2}}\hat{A}_t - \frac{r_2}{\lambda_2}\frac{1}{1-\frac{\rho}{\lambda_2}}\hat{A}_t \\
&= -\frac{\rho r_1 + r_2}{\lambda_2 - \rho}\hat{A}_t.
\end{aligned} \tag{93}$$

Solution Method 1 (continued):

- since

$$\begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} \equiv P^{-1} \begin{pmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{pmatrix}, \tag{94}$$

Solution Method 1 (continued):

- since

$$\begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} \equiv P^{-1} \begin{pmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{pmatrix}, \tag{94}$$

- we have

$$x_{2t} = p_{21}\hat{k}_t + p_{22}\hat{\lambda}_t.$$

Thus, the equilibrium path for λ_t is given by

$$\hat{\lambda}_t = -\frac{p_{21}}{p_{22}}\hat{k}_t - \frac{1}{p_{22}}\frac{\rho r_1 + r_2}{\lambda_2 - \rho}\hat{A}_t. \tag{95}$$

Solution Method 1 (continued):

- Once we obtain $\hat{\lambda}_t$, we have

$$\begin{pmatrix} \hat{c}_t \\ \hat{n}_t \end{pmatrix} = \Pi \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \end{pmatrix} \tag{96}$$

Solution Method 1 (continued):

- Once we obtain $\hat{\lambda}_t$, we have

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- and

$$\hat{k}_{t+1} = \pi \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \end{pmatrix}. \tag{97}$$

Solution Method 2

- We have

$$\begin{pmatrix} \hat{c}_t \\ \hat{n}_t \end{pmatrix} = A_{2\times 3} \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \\ \hat{\lambda}_t \end{pmatrix} \tag{98}$$

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- and

$$E_t \begin{pmatrix} \hat{k}_{t+1} \\ \hat{A}_{t+1} \\ \hat{\lambda}_{t+1} \end{pmatrix} = B_{3\times 3} \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \\ \hat{\lambda}_t \end{pmatrix} = P\Lambda P^{-1} \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \\ \hat{\lambda}_t \end{pmatrix}. \tag{99}$$

Solution Method 2 (continued)

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$$x_t = P^{-1} \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \\ \hat{\lambda}_t \end{pmatrix}, \tag{100}$$

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- and suppose $|\lambda_3| > 1$. Then a stationary solution is given by

$$x_{3t} = 0, \qquad (101)$$

Solution Method 2 (continued)

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- or

$$\hat{\lambda}_t = -\frac{p_{31}}{p_{33}}\hat{k}_t - \frac{p_{32}}{p_{33}}\hat{A}_t. \tag{103}$$

An example

- Suppose we have

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} = QE_t \begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} \tag{104}$$

  Where $\hat{c}_t$ is the endogenous variable and $\hat{k}_t$ is the state variable. The matrix Q is:

$$Q = \begin{bmatrix} -2.5 & -9 \\ 1.5 & 5 \end{bmatrix} \tag{105}$$

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$$Q = \left[\begin{array}{cc} -2.5 & -9 \\ 1.5 & 5 \end{array}\right] \tag{105}$$

- Solve $\hat{c}_t$ in term of $\hat{k}_t$.

An example

- Follow the procedure in class, The system can be written as :

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} \tag{106}$$

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- Therefore we have

$$\hat{c}_t + 2\hat{k}_t = 0 \tag{107}$$

or

$$\hat{c}_t = -2\hat{k}_t \tag{108}$$

Calibration

- There are 5 structural parameters in the model, $\{\alpha, \beta, \delta, \gamma, \rho\}$,so we need five conditions

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- In the steady-state

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\begin{aligned}
1 &= \beta\left(1+r^*\right) \\
\alpha &= \frac{\left(r^*+\delta\right)k^*}{y^*}; 1-\alpha = \frac{wn^*}{y^*} \\
\frac{k^*}{y^*} &= \frac{\alpha\beta}{1-\beta\left(1-\delta\right)}.
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\end{aligned}
$$

- If $r = 4\%$ a year (i.e., 1% a quarter), then in a quarterly model we have $\beta = \frac{1}{1.01} = 0.99$.

Calibration (continued)

- $\frac{wn^*}{y^*} = 1 - \alpha$; and $\frac{wn^*}{y^*} = 0.65$ in data, so we have $\alpha = 0.35$.

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- $\frac{k^*}{y^*} = 10$ in data, by $\alpha \frac{y}{k} = (r + \delta)$, we have

$$\delta = \frac{\alpha}{k/y} - r = 0.035 - 0.01 = 0.025. \tag{109}$$

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- $\frac{wn^*}{y^*} = 1 - \alpha$; and $\frac{wn^*}{y^*} = 0.65$ in data, so we have $\alpha = 0.35$.
- $\frac{k^*}{y^*} = 10$ in data, by $\alpha \frac{y}{k} = (r + \delta)$, we have

$$\delta = \frac{\alpha}{k/y} - r = 0.035 - 0.01 = 0.025. \tag{109}$$

- implies a steady-state rate of saving equal to

$$s^* = \delta \frac{k}{y} = 0.25 = 25\%, \tag{110}$$

and a steady-state $\frac{c}{y} = 1 - s = 0.75$.

Calibration (continued)

- Also, we know that the fraction of hours worked each week is about $n^* = \frac{40}{24\times 7} \approx .2\dot{4}$ (which implies that the fraction of hours worked in a quarter is also 0.24). Given the steady-state relationship,

$$\frac{\gamma n}{1-n} = (1-\alpha)\frac{y}{c} = \frac{0.76}{0.24} \times \frac{0.65}{0.75} = 2.74, \tag{111}$$

Calibration (continued)

- Also, we know that the fraction of hours worked each week is about $n^* = \frac{40}{24 \times 7} \approx .2\dot{4}$ (which implies that the fraction of hours worked in a quarter is also 0.24). Given the steady-state relationship,

$$\frac{\gamma n}{1-n} = (1-\alpha)\frac{y}{c} = \frac{0.76}{0.24} \times \frac{0.65}{0.75} = 2.74, \tag{111}$$

- Finally, to calibrate ρ, we can estimate the Solow residual using

$$\hat{A}_t = \hat{y}_t - \alpha \hat{k}_t - (1-\alpha)\hat{n}_t,$$

and then estimate ρ by

$$\hat{A}_t = \rho \hat{A}_{t-1} + e_t. \tag{112}$$

Simulation

- The decisions rules can be arranged into:

$$\begin{pmatrix} \hat{c}_t \\ \hat{n}_t \\ \hat{y}_t \\ \vdots \end{pmatrix} = \Pi \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \end{pmatrix} \tag{113}$$

$$\begin{pmatrix} \hat{k}_t \\ \hat{A}_t \end{pmatrix} = M \begin{pmatrix} \hat{k}_{t-1} \\ \hat{A}_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_t . \tag{114}$$

Simulation (continued)

- Starting from the steady state at $t=0$, we have $\hat{k}_{0-j}=0$ and $\hat{A}_{0-j}=0$ for $j>0$. Given the sequence, $\{\varepsilon_t\}_{t=0}^{T}$ (drawn from a random generator), (8) implies

$$\begin{pmatrix} \hat{k}_0 \\ \hat{A}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_0 \tag{115}$$

$$\begin{pmatrix} \hat{k}_1 \\ \hat{A}_1 \end{pmatrix} = M \begin{pmatrix} \hat{k}_0 \\ \hat{A}_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_1 \tag{116}$$

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{A}_{t+1} \end{pmatrix} = M \begin{pmatrix} \hat{k}_{t-1} \\ \hat{A}_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_{t+1}. \tag{117}$$

Substituting the generated sequences, $\left\{\hat{k}_t, \hat{A}_t\right\}_{t=0}^{T}$, into (113) produces the sequences $\left\{\hat{c}_t, \hat{n}_t, \hat{y}_t, \ldots\right\}_{t=0}^{T}$.