Notes on dynamic programing

Pengfei Wang

Hong Kong University of Science and Technology

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The problem

- We are interested in the following problem

$$\max_{\{x_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad \text{(SP)}$$

s.t

$$x_{t+1} \in \Gamma(x_t), t = 0, 1, 2, .., \qquad (1)$$

$$x_0 \in X \text{ given} \qquad (2)$$

The functional equation presentation of the the problem

- We can present the above (SP) in the (FE) form

$$v(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\} \tag{FE}$$

for all $x \in X$

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- The solution v to (FE) evaluated at x_0, gives the maximum value in (SP) if and only if it satisfies

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- Richard Bellman called this idea the Principle of Optimality.

An example

- We are interested in solving

$$\max_{\{c_t, s_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{4}$$

s.t

$$c_t + s_{t+1} \leq Rs_t \tag{5}$$

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$$(c_t, s_{t+1}) \geq 0 \tag{6}$$

- Use s_{t+1} to substitute c_t out, we have

$$\max_{\{s_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \log(Rs_t - s_{t+1}) \tag{7}$$

s.t

$$0 \leq s_{t+1} \leq Rs_t \tag{8}$$

Principle of Optimality -proof : Notations

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$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t), t = 0, 1, 2, ...\} \tag{9}$$

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- Let $\tilde{x} = (x_0, x_1, x_2, ...)$ denote a typical element $\Pi(x_0)$,we assume $\Gamma(x)$ is noempty for all $x \in X$. and
- for all $x_0 \in X$ and $\tilde{x} \in \Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists

Principle of Optimality -Notations (continued)

futher notation

- The partial sum

$$u_n(\tilde{x}) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) \tag{10}$$

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- ans suppose function

$$v^*(x_0) = \max u(\tilde{x}), \tag{vstar}$$

where $\tilde{x} \in \Pi(x_0)$ is feasible plan.

Principle of Optimality (continued)

Our interests is estalbish the connections between v defined by (FE) and (vstar). Notice that $v^*(x_0)$ is always unique

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- proof:

$$\begin{aligned} u(\tilde{x}) &= \lim_{n\to\infty}\sum_{t=0}^{n}\beta^t F(x_t, x_{t+1}) \\ &= F(x_0, x_1) + \beta \lim_{n\to\infty}\sum_{t=0}^{n}\beta^t F(x_{t+1}, x_{t+2}) \\ &= F(x_0, x_1) + \beta u(\tilde{x}') \qquad (13) \end{aligned}$$

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- Now prove that

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- step 2: we proove

$$v^*(x_0) \geq \max\{F(x_0, x_1) + \beta v^*(x_1)\}$$

Principle of Optimality –proof of step 2

- Notice by definition we have

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- choosing $\Pi(x_1)$, such $v^*(x_1) = \max u(\tilde{x}')$, so we must have

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- again it holds for any x_1, so we have

$$v^*(x_0) \geq \max\{F(x_0, x_1) + \beta v^*(x_1)\} \quad (18)$$

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- again by definition $v^*(x_0) = \max u(\tilde{x})$,where $\tilde{x} \in \Pi(x_0)$, suppose $v^*(x_0) = u(\tilde{x}^*)$, we then have

$$v^*(x_0) = F(x_0, x_1^*) + \beta u(\tilde{x}^{*'}) \tag{20}$$

where $\tilde{x}^{*'} = [x_1^*, x_2^*, ...]$

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where $\tilde{x}^{*'} = [x_1^*, x_2^*, ...]$

- by definition of v^*, we have $u(\tilde{x}^{*'}) \leq v^*(x_1^*)$, so we have

$$\begin{aligned} v^*(x_0) &\leq F(x_0, x_1^*) + \beta v^*(x_1^*) \\ &\leq \max\{F(x_0, x_1) + \beta v^*(x_1)\} \end{aligned} \tag{21}$$

The second line comes from the definition of max

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- step 4, we assume $\lim_{n\to\infty} \beta^n v(x_n) = 0$ and proove

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$$\begin{aligned} v(x_0) &\geq F(x_0, x_1) + \beta v(x_1) \\ &\geq F(x_0, x_1) + \beta[F(x_1, x_2) + \beta v(x_2)] \\ &\geq u_n(\tilde{x}) + \beta^{n+1} v(x_{n+1}) \end{aligned} \tag{23}$$

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- take limit we have $v(x_0) \geq u(\tilde{x})$ for $\tilde{x} \in \Pi(x_0)$.

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- taking limit we have $v(x_0) \geq u(\tilde{x})$ for $\tilde{x} \in \Pi(x_0)$., so we have $v(x_0) \geq \max u(\tilde{x}) = v^*(x_0)$

Principle of Optimality -proof of step 4

- suppose $x_{t+1} = \tilde{\Gamma}(x_t)$, solves

$$v(x_t) = \max\{F(x_t, x_{t+1}) + \beta v(x_{t+1})\} \tag{26}$$

- where $\hat{x}_{t+1} = \tilde{\Gamma}(\hat{x}_t)$,

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Principle of Optimality -proof of step 4

- so we have

$$\begin{aligned} v(x_0) &= F(x_0,\hat{x}_1)+\beta v(\hat{x}_1) \\ &= F(x_0,\hat{x}_1)+\beta F(\hat{x}_1,\hat{x}_2)+\beta^2 v(\hat{x}_2) \\ &= \ldots \\ &= u_n(\hat{x})+\beta^{n+1} v(\hat{x}_{n+1}) \end{aligned} \tag{28}$$

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- Notice $\hat{x} \in \Pi(x_0)$, so we have $v(x_0) \leq v^*(x_0)$ by definition. We now have

$$v = v^* \tag{30}$$

First Order Conditions

- Once we know

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- together we have

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Euler Equations

- There is a classical solution to the (SP) problem

$$\max_{\{x_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{SP}$$

s.t

$$x_{t+1} \in \Gamma(x_t), t = 0, 1, 2, .., \tag{40}$$

$$x_0 \in X \text{ given} \tag{41}$$

Euler Equations-Necessary Conditions

- Consider $\{x_{t+1}^*\}_{t=0}^{\infty}$ solves the problem SP, given x_0, then for $t = 0, 1, ... x_{t+1}^*$ must solve

$$\max_y F(x_t^*, y) + \beta F(y, x_{t+2}^*) \tag{42}$$

st

$$y \in \Gamma(x_t^*) \text{ and } x_{t+2}^* \in \Gamma(y) \tag{43}$$

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$$0 = \frac{\partial F(x^*_t, x^*_{t+1})}{\partial x_{t+1}} + \beta \frac{\partial F(x^*_{t+1}, x^*_{t+2})}{\partial x_{t+1}} \tag{44}$$

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- An addtionaly boundary condtions are supplied by the transversality condition

$$\lim_{t\to\infty} \frac{\partial F(x^*_t, x^*_{t+1})}{\partial x_t} x^*_t = 0 \tag{45}$$

Euler Equations-Sufficience Conditions: additional assumptions

- Denote $F_x(x_t, x_{t+1})$ as the partial derivitative with respect to first argument and $F_y(x_t, x_{t+1})$ as second partial derivitative with respect to second argument.

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- F is continous, concave, and differentiable.

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- Assume $F_x(x_t, x_{t+1}) \geq 0$, and assume the feasible allocation $x_t \geq 0$.
- F is continous, concave, and differentiable.
- if equations (44) and (45) satisfied, then $\{x^*_{t+1}\}_{t=0}^{\infty}$ solves (SP) problem.

Euler Equations-Sufficience Conditions-proof

- Denote the distance

$$D = \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t [F(x_t^*, x_{t+1}^*) - F(x_t, x_{t+1})] \tag{46}$$

where $(\tilde{x}_t^*, x_t) \in \Pi(x_0)$ both are feasible plans.

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- By F is concave, we have

$$F(x_t, x_{t+1}) \leq F_x(x_t^*, x_{t+1}^*)(x_t - x_t^*) + F_y(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*) \tag{47}$$

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- so we have

$$D \geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F_x(x_t^*, x_{t+1}^*)(x_t^* - x_t) + F_y(x_t^*, x_{t+1}^*)(x_{t+1}^* - x_{t+1})] \tag{48}$$

Euler Equations-Sufficience Conditions-proof (continued)

- We already have

$$\begin{aligned} D &\geq \lim_{T\to\infty}\sum_{t=0}^{T}\beta^t[F_x(x_t^*,x_{t+1}^*)(x_t^*-x_t)+F_y(x_t^*,x_{t+1}^*)(x_{t+1}^*-x_{t+1})] \\ &= F_x(x_0^*,x_1^*)(x_0^*-x_0)+F_y(x_0^*,x_1^*)(x_1^*-x_1) \\ &\quad +\beta F_x(x_1^*,x_2^*)(x_1^*-x_1)+\beta F_y(x_1^*,x_2^*)(x_2^*-x_2) \\ &\quad +.. \\ &\quad +\beta^T F_x(x_T^*,x_{T+1}^*)(x_T^*-x_T) \\ &\quad +\beta^T F_y(x_T^*,x_{T+1}^*)(x_{T+1}^*-x_{T+1}) \end{aligned}$$

Euler Equations-Sufficience Conditions-proof (continued)

- By

$$0 = F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*) \quad (49)$$

$$x_0 = x_0^* \quad (50)$$

Euler Equations-Sufficience Conditions-proof (continued)

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$$0 = F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*) \quad (49)$$

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- we have

$$D \geq \lim_{T\to\infty} \beta^T F_y(x_T^*, x_{T+1}^*)(x_{T+1}^* - x_{T+1}) \quad (51)$$

Euler Equations-Sufficience Conditions-proof (continued)

- again using

$$F_y(x_T^*, x_{T+1}^*) + \beta F_x(x_{T+1}^*, x_{T+2}^*) = 0 \tag{52}$$

Euler Equations-Sufficience Conditions-proof (continued)

- again using

$$F_y(x_T^*, x_{T+1}^*) + \beta F_x(x_{T+1}^*, x_{T+2}^*) = 0 \tag{52}$$

- we have

$$\begin{aligned} D &\geq -\lim_{T\to\infty} \beta^{T+1} F_x(x_{T+1}^*, x_{T+2}^*)(x_{T+1}^* - x_{T+1}) \\ &= -\lim_{T\to\infty} \beta^T F_x(x_T^*, x_T^*)(x_T^* - x_T) \end{aligned} \tag{53}$$

Euler Equations-Sufficience Conditions-proof (continued)

- again using

$$F_y(x^*_T, x^*_{T+1}) + \beta F_x(x^*_{T+1}, x^*_{T+2}) = 0 \tag{52}$$

- we have

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- Since $F_x(x^*_T, x^*_T) \geq 0; x_T \geq 0$, so we have

$$D \geq -\lim_{T\to\infty} \beta^{T} F_x(x^*_{T}, x^*_{T}) x^*_T = 0 \tag{54}$$

Back to example

- Use s_{t+1} to substitute c_t out, we have

$$\max_{\{s_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \log(Rs_t - s_{t+1}) \tag{55}$$

s.t

$$0 \leq s_{t+1} \leq Rs_t \tag{56}$$

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$$0 \le s_{t+1} \le Rs_t \tag{56}$$

- Define it recursively

$$v(s_t) = \max\{\log(Rs_t - s_{t+1}) + \beta v(s_{t+1})\} \tag{57}$$

Back to example-FOC

- with respect to s_{t+1}

$$\frac{1}{Rs_t - s_{t+1}} = \beta v'(s_{t+1}) \tag{58}$$

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- the transversality condition;

$$\lim \beta^t \frac{1}{c_t} s_t = 0 \tag{61}$$

Uncertainty

- The problem changes to

$$\max_{\{x_{t+1}\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}, z_t) \tag{SP}$$

s.t

$$x_{t+1} \in \Gamma(x_t, z_t), t = 0, 1, 2, .., \tag{62}$$

$$x_0 \in X \ ; z_0 \in Z \text{ given} \tag{63}$$

where z_t are exogenous shocks.

Uncertainty-Principle of Optimality

- Like the certainty case, we can present the this problem into

$$v(s_t) = v(x_t, z_t) = \max_{x_{t+1} \in \Gamma(x_t, z_t)} \{F(x_t, x_{t+1}, z_t) + \beta E_t v(x_{t+1}, z_{t+1})\} \tag{64}$$

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$$F_y(x_t, x_{t+1}, z_t) + \beta E_t v_x(x_{t+1}, z_{t+1}) = 0 \tag{65}$$

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- Envelop Theory

$$v_x(x_t, z_t) = F_x(x_t, x_{t+1}, z_{t+1}) \tag{66}$$

- Together implies

$$F_y(x_t, x_{t+1}, z_t) + \beta E_t F_x(x_{t+1}, x_{t+2}, z_{t+1}) = 0 \tag{67}$$

Uncertainty-Principle of Optimality

- The first order condition define a policy function

$$x_{t+1} = G(x_t, z_t) \tag{68}$$

Uncertainty-Principle of Optimality

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- Such that

$$\begin{aligned} G(x_t, z_t) &= \{x_{t+1} \in \Gamma(x_t, z_t) : \\ v(x_t, z_t) &= F(x_t, y_t, z_{t+1}) + \beta E_t v(x_{t+1}, z_{t+1})\} \end{aligned} \tag{69}$$

Uncertainty-Euler Equation

- Like the certainty case, the necessary condition should solves

$$\max_{x_{t+1}}\{F(x_t^*, x_{t+1}, z_t) + \beta E_t F(x_{t+1}, x_{t+2}^*, z_{t+1})\} \tag{70}$$

where $x_{t+2}^* \in G(x_{t+1}, z_{t+1})$.

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where $x_{t+2}^* \in G(x_{t+1}, z_{t+1})$.

- Euler equations implies

$$F_y(x_t^*, x_{t+1}^*, z_t) + \beta E_t F_x(x_{t+1}^*, x_{t+2}^*, z_{t+1}) = 0 \tag{71}$$

Uncertainty-Example

- Now suppose R_t are stochastic

$$\max_{\{s_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \log(R_t s_t - s_{t+1}) \tag{72}$$

s.t

$$0 \leq s_{t+1} \leq R_t s_t \tag{73}$$

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$$0 \leq s_{t+1} \leq R_t s_t \tag{73}$$

- The Euler equations are

$$\frac{1}{R_t s_t - s_{t+1}} = \beta E_t \frac{R_{t+1}}{R_{t+1} s_{t+1} - s_{t+2}} \tag{74}$$

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- or

$$\frac{1}{c_t} = \beta E_t \frac{R_{t+1}}{c_{t+1}} \tag{75}$$