Notes on endogenous growth models

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- We start with an AK model
- Then we introduce the Romer model and its variants

Household

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$$U = \int_0^\infty e^{-\rho t} u(C_t) L_t dt. \tag{1}$$

Where C_t is the consumption per capita and L_t is the total population of the economy at time t.

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- The household maximize (1) , with the constraint

$$L_t C_t + \dot{K}_t = F(K_t) - \delta K_t = AK_t - \delta K_t \qquad (3)$$

Transformation

- The life utility of the household is thus

$$U = L_0 \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} e^{nt} dt = L_0 \int_0^\infty e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} \qquad (4)$$

Where $\beta \equiv \rho - n$. We assume $\beta > 0$. Without loss of generality, L_0 is assumed to be 1.

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$$c_t + \dot{k}_t = Ak_t - (\delta + n)k_t \tag{7}$$

First order conditions

- The problem can solved by the caculus of variations method by defining:

$$\begin{aligned} J &= \int_0^\infty e^{-\beta t} \left\{ \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t [Ak_t - (\delta + n)k_t - c_t - \dot{k}_t] \right\} dt \\ &= \int_0^\infty e^{-\beta t} H(x_t, \dot{x}_t) dt \end{aligned}$$

Where $x_t = [c_t, \lambda_t, k_t]$.

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&= \int_0^\infty e^{-\beta t} H(x_t, \dot{x}_t) dt
\end{aligned}
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  Where $x_t = [c_t, \lambda_t, k_t]$.

- Therefore the necessary condition

$$
\frac{\partial}{\partial x_t} e^{-\beta t} H(x_t, \dot{x}_t) = \frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} e^{-\beta t} H(x_t, \dot{x}_t) \right] \tag{8}
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$$\begin{aligned} e^{-\beta t}\lambda_t[A-(\delta+n)] &= -\dot{\lambda}_t e^{-\beta t}+\beta\lambda_t e^{-\beta t} \\ A-(\delta+n) &= -\frac{\dot{\lambda}_t}{\lambda_t}+\beta \end{aligned} \tag{14}$$

First order conditions (continued)

- Since $\frac{\dot{\lambda}_t}{\lambda_t} = -\theta \frac{\dot{c}_t}{c_t}$. So we have two equation :

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- and resource constraint implies

$$\dot{k}_t = Ak_t - (\delta + n)k_t - c_t \quad (17)$$

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- We assume that $A > (\delta + \rho)$.So Consumption growth at a constant rate, so we have

$$c_\tau = c_t e^{\frac{(A-\delta-\rho)(\tau-t)}{\theta}} \tag{18}$$

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$$\frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = g = A - \delta - \rho \tag{22}$$

Interpretation 1

- K includes both physical and human capital. Suppose that there is no poupulation growth rate and the household maximizes:

$$\max_{c_t, \dot{k}_t} \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \tag{23}$$

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- s.t

$$c_t + \dot{k}_t + \dot{h}_t = F(k_t, h_t) - \delta_k k_t - \delta_h h_t \tag{24}$$

First Order Condition

- first order conditions w.r.t k_t, h_t

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[\frac{\partial F}{\partial k_t} - \delta_k - \rho] \tag{25}$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[\frac{\partial F}{\partial h_t} - \delta_h - \rho] \tag{26}$$

$$Y_t = F(k_t, h_t) = k_t f(h_t / k_t) \tag{27}$$

First Order Condition

- so we have

$$\frac{\partial F}{\partial k_t} = f\left(h_t / k_t\right) - \left(h_t / k_t\right) f'\left(h_t / k_t\right)$$

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- so we have

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- This implies

$$f(h_t/k_t) - (h_t/k_t)\, f'(h_t/k_t) - \delta_k = f'(h_t/k_t) - \delta_h$$

The above equation implies $\frac{h_t}{k_t} = constant$. Define $f(h_t/k_t) = A$, we then have a AK model.

Spillover

- Consider a model in which each households solves

$$\max_{c_t, \dot{k}_t} \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \tag{29}$$

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- where K_t is the aggregate capital. We assume

$$F(k_t, K_t) = A k_t^{\alpha} K_t^{1-\alpha} \tag{31}$$

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- The growth rate in this case is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[\alpha A - (\rho + \delta)] = g$$

Taken K_t as given and in equilibrium $k_t = K_t$.

Introduction

- An endogenous AK model;
- Production has two level: final good and intermediate goods;
- the number of intermediate goods is expanding over time due to R&D activity;
- The new invention is titled to monopoly power.

Household

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$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma} - 1}{1 - \sigma} \tag{32}$$

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- From the above equation we have that

$$g = [\beta(1+r)]^{\frac{1}{\sigma}} - 1 \tag{34}$$

Production Sectors: the final goods producer

- Consider first the problem of the final goods producers. Their production function is given by

$$Y = H_Y^{\alpha} L^{\gamma} \int_0^N x_i^{1-\alpha-\gamma} di \tag{35}$$

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- where H_Y is human capital devoted to final goods production, L is labor, N is the total number of intermediate goods currently in existence and x_i is the quantity of the intermediate good.

Production Sectors: the final goods producer

- Final goods produce choose input to maximize their profits and ,therefore, solve

$$\max_{H_Y, L, \{x_i\}} H_Y^{\alpha} L^{\gamma} \int_0^N x_i^{1-\alpha-\gamma} di - w_H H_Y - w_L L - \int_0^N p_i x_i di \qquad (36)$$

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- The price of intermediate goods i is hence

$$p_i = \frac{\eta r}{1-\alpha-\gamma} \tag{42}$$

Production Sectors: the intermediate goods producer

- And its profit $\pi_i = (p_i - \eta r)\, x_i$ equals to

$$\begin{aligned} \pi_i &= (\alpha + \gamma) p_i x_i \\ &= (\alpha + \gamma)(1 - \alpha - \gamma) H_Y^{\alpha} L^{\gamma} x_i^{1-\alpha-\gamma} \end{aligned} \tag{43}$$

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$$= (\alpha+\gamma)(1-\alpha-\gamma) H_Y^{\alpha} L^{\gamma} x_i^{1-\alpha-\gamma}$$

- The value of the right to produce an intermediate goods must equal to the discounted profit

$$V_t = \pi_t + \frac{\pi_{t+1}}{1+r_{t+1}} + \frac{\pi_{t+2}}{(1+r_{t+1})(1+r_{t+2})} + \ldots \tag{44}$$

Production Sectors: Research firms

- Given an input of human capital H_N and the stock of technology N, a firm can produce $\delta H_N N$ new varieties. Hence the stock of technology evolves according to

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- wage is

$$P_{Nt} \delta N_t = w_{Ht} \tag{47}$$

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- The value of each intermediate good firm is constant.
- Consumption, K grow at the same rate rate

Balanced growth path (continued)

- The value of intermediate good firm

$$V = P_N = \frac{(\alpha+\gamma)(1-\alpha-\gamma)H_Y^{\alpha}L^{\gamma}x^{1-\alpha-\gamma}}{r} \tag{48}$$

Balanced growth path (continued)

- The value of intermediate good firm

$$V = P_N = \frac{(\alpha+\gamma)(1-\alpha-\gamma)H_Y^{\alpha}L^{\gamma}x^{1-\alpha-\gamma}}{r} \tag{48}$$

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- The wage of human capital in the final goods sector

$$\begin{aligned} w_H &= \alpha H_Y^{\alpha-1} L^{\gamma} \int_0^N x_i^{1-\alpha-\gamma} di \\ &= \alpha H_Y^{\alpha-1} L^{\gamma} N x^{1-\alpha-\gamma} \end{aligned} \tag{50}$$

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- This implies

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$$H_Y = \frac{\alpha r}{\delta(1-\alpha-\gamma)(\alpha+\gamma)} \tag{52}$$

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- together

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Growth through Product Quality Improvement

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- Growth in this model is based on quality improvement but the number of goods is fixed.

Model Element

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- Production of final goods

$$Y_t = \exp \int_0^1 \log q_t(i) y_t(i) di \tag{57}$$

Model Element

- Standard household. Assuming no population growth. Time is discret and the presentative consumer's preference is

$$U = \sum_{t=0}^{\infty} \beta^t \log(C_t) \tag{56}$$

- Production of final goods

$$Y_t = \exp \int_0^1 \log q_t(i) y_t(i) di \tag{57}$$

- $q_t(i)$ is the quality of each goods. The quality $q_t(i) = \lambda^{m_t(i)} q_0$ defines the state of art in industry i

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- It is useful to define the effective output in industry as

$$\tilde{y}(i) = q_t(i) y_t(i), \tag{58}$$

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- And the production function to produce the effective output is:

$$\tilde{y}(i) = q_t(i)n(i). \tag{60}$$

The incumbent intermediate firm

- The incumbent intermediate producer's face the following demand curve :

$$\tilde{y}_t(i) = \left\{ \begin{array}{cc} \frac{Y_t}{p_t(i)} & \text{if } p_t(i) \leq \frac{w_t}{\lambda^{m_t(i)-1} q_0} \\ 0 & \text{otherwise} \end{array} \right\} \tag{61}$$

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- and the profit is

$$\begin{aligned} \pi(t) &= p(i)y(i) - \frac{1}{\lambda} p(i)y(i) \\ &= \{1 - \frac{1}{\lambda}\} Y_t \end{aligned} \tag{67}$$

The Inventor

- A inventor can spend $a\phi$ unit labor to do research, it suceed with ϕ probability to improve the current technology from λ^{m_t} to λ^{m_t+1} and hence it can replace the incumbent producer, his expected payoff

$$-a\phi w_t + \phi V_t \tag{68}$$

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$$\pi_{t+j} = Y_{t+j}\{1 - \frac{1}{\lambda}\} \tag{70}$$

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V_t &= \sum_{j=0}^{\infty} q_j \beta^j \frac{C_t}{C_{t+j}} Y_{t+j}\{1-\frac{1}{\lambda}\} \\
&= \sum_{j=0}^{\infty} \beta^j (1-\phi)^j_t C_t \{1-\frac{1}{\lambda}\} \qquad (71)
\end{aligned}
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$$= \sum_{j=0}^{\infty} \beta^j (1-\phi)_t^j C_t \{1-\frac{1}{\lambda}\} \quad (71)$$

- by $Y_t = C_t$ we have

$$V_t = \frac{Y_t(1-\frac{1}{\lambda})}{1-\beta(1-\phi)} \quad (72)$$

Equilibrium

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- This leads to

$$\frac{\dot{Y}_t}{Y_t} = \phi \log \lambda \tag{83}$$