Notes on the OLG Model

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Introduction?

- Introduced by Samuelson (1958).
- It contains agents who are born at diffrent dates and have finite lifetimes, even though the economy goes on forever.
    - competitive equilibria in the OLG model may not to be Pareto optimal
    - A closely related feature of the model is that it has a role for fiat money (Bubble)

The Basic Model

- Suppose that $t = 1, 2, ...,$ and that at every date t there is born a new generation G_t of individuals who live for two periods.
- There is also a generation G_0 around at $t = 1$ who only live for one period, called the "initial old."
- every generation consists of a [0, 1] continuum of homogeneous agents.

The Basic Model (continued)

- Let c_{1t} and c_{2t+1} denote consumption of an individual from G_t, $t \geq 1$, in the 1st and 2nd periods of life
- Let e_1 and e_2 denote his (time-invariant) endowments in the 1st and 2nd periods of life.
- His utility function $u(c_{1t}, c_{2t+1})$ is strictly increasing and quasi-concave.
- Members of generation G_0 consume only c_{21} and are endowed with only e_2.

Recursive Competitive Equilibrium (RCE)

- Let s_t denote savings or loans by a member of G_t at t, and R_t the gross (principal plus interest) return on savings between t and $t+1$.
- for $t \geq 1$, a member of G_t at t solves

$$\max u(c_{1t}, c_{2t+1}) \tag{1}$$

with the constraints

$$c_{1t} = e_1 - s_t, \tag{2}$$

and

$$c_{2t+1} = e_2 + R_t s_t, \tag{3}$$

and $(c_{1t}, c_{2t+1}) \geq 0$.

RCE (Continued)

- A RCE is a sequence $\{R_t, c_{1t}; c_{2t+1}, s_t\}$ such that: $c_{21} = e_2$; given $\{R_t\}$,$\{c_{1t}, c_{2t+1}, s_t\}$ solves the maximization problem of G_t for all $t \geq 1$; and the market clears in every period.
- market clearing condition

$$c_{1t} + c_{2t} = e_1 + e_2. \tag{4}$$

RCE (Continued)

Lemma

the only equilibrium allocation here is autarchy, namely $(c_{1t}, c_{2t+1}) = (e_1, e_2)$ *for all* t.

Proof.

To verify this, first note that homogeneity implies no trade within a generation. Then note that in any equilibrium $c_{21} = e_2$, *and combined with market clearing this implies* $c_{11} = e_1$. *Then by equation (2) and (3) imply* $s_1 = 0$ *and* $c_{22} = e_2$. *Using the market clearing condition in period 2, we have* $c_{12} = e_1$. *Repeat the above steps, we conclude that* $(c_{1t}, c_{2t+1}) = (e_1, e_2)$ *for all* t □

Efficience of RCE

An interesting property of the OLG model is that equilibria may not be Pareto optimal.

Example

For example, suppose $(e_1, e_2) = (1, 0)$ and $u(c_{1t}, c_{2t+1}) = c_{1t} + c_{2t+1}$, for all $t \geq 1$ (this example may seem special because the indifference curves are linear, but it will be clear below that the point is general). Then the autarchy allocation is Pareto dominated by $(c_{1t}, c_{2t+1}) = (0, 1)$ for all t.

Efficience of RCE (continued)

Let μ be the marginal rate of substitution function

$$\mu = \frac{u_1(c_1, c_2)}{u_2(c_1, c_2)} = R, \tag{5}$$

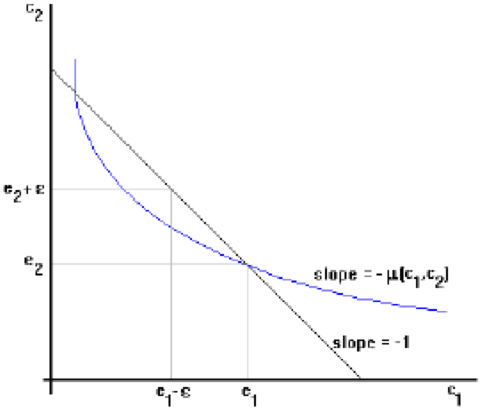
we have the following Lemma.

Lemma

with strictly convex indiffence curves, the unique equilibrium allocation is Pareto optimal if and only if the marginal rate of substitution at the endowment point is bigger than unity,namely $\mu(e_1, e_2) \geq 1$.

Efficience of RCE (continued)

First consider $\mu(e_1, e_2) < 1$,



Efficience of RCE (continued)

In the case $\mu(e_1, e_2) < 1$, to show autarchy is inefficient, consider the alternative allocation $(c_{1t}, c_{2t+1}){=}(e_1 - \varepsilon, e_2 + \varepsilon)$ for all $t \geq 1$ and $c_{21} = e_2 + \varepsilon$, where $\varepsilon \in (0, e_1]$

- The alternative allocation is feasible. The market clearing condition $c_{1t} + c_{2t} = e_1 + e_2$ holds.
- The "initial old" is strictly better off. $c_{21} = e_2 + \varepsilon > e_2$.
- The member in G_t, we have

$$\begin{aligned} \Delta u &= u(e_1 - \varepsilon, e_2 + \varepsilon) - u(e_1, e_2) \qquad (6) \\ &\simeq -\varepsilon u_1(e_1, e_2) + \varepsilon u_2(e_1, e_2) \end{aligned}$$

since $\frac{u_1(e_1,e_2)}{u_2(e_1,e_2)} = \mu < 1$, we have

$$\Delta u \simeq u_2(e_1, e_2)\varepsilon[1 - \mu] > 0 \qquad (7)$$

so member in G_t is strictly better off too.

Efficience of RCE (continued)

In the case $\mu(e_1, e_2) \geq 1$, we use contradiction to show autarchy is efficient. Consider the $(c_{1t}, c_{2t+1})= (e_1, e_2)$ and $c_{12} = e_2$ is not pareto optimal. We now consider a alternative allocation, such that

$$\tilde{c}_{21} = e_2 + \varepsilon_1 \tag{8}$$

and $\{\tilde{c}_{1t}, \tilde{c}_{2t+1}\} = \{e_1 - \varepsilon_t, e_2 + \varepsilon_{t+1}\}$ for $t \geq 1$.

- To make the "initial old " better off, we must have $\varepsilon_1 \geq 0$.
- To make the member in G_t at least as good as before, we must have

$$\Delta u \simeq -\varepsilon_t u_1(e_1, e_2) + \varepsilon_t u_2(e_1, e_2)$$

- The "initial old" is strictly better off. $c_{21} = e_2 + \varepsilon_1 > e_2$.

Efficience of RCE (continued)

- The member in G_t, we have

$$\begin{aligned}\Delta u &= u(e_1-\varepsilon, e_2+\varepsilon) - u(e_1, e_2) \\ &\simeq -\varepsilon_t u_1(e_1, e_2) + \varepsilon_{t+1} u_2(e_1, e_2)\end{aligned} \quad (9)$$

because $\frac{u_1(e_1,e_2)}{u_2(e_1,e_2)} = \mu \geq 1$, we have $\Delta u \geq 0$ requires

$$\varepsilon_{t+1} u_2(e_1, e_2) \geq \varepsilon_t u_1(e_1, e_2) \quad (10)$$

or

$$\varepsilon_{t+1} \geq \varepsilon_t \frac{u_1(e_1, e_2)}{u_2(e_1, e_2)} = \mu \varepsilon_t \quad (11)$$

For any $\varepsilon_1 > 0$, this requires $\lim_{t\to\infty} \varepsilon_{t+1} = \infty$ and which becomes infeasible in a finite period. So we must $\varepsilon_t = \varepsilon_1 = 0$.

Money

Into the model described above, we now introduce a constant amount M of fiat money, held in period 1 by the initial old G_0. By definition, fiat money is an object that has no intrinsic (consumption) value, but could potentially have exchange value.

- Let q_t be the value of money at date t. If $q_1 > 0$ then the initial old can consume $c_{21} = e_2 + q_1 M > e_2$ without violating their budget constraint.

Member in G

t

- for $t \geq 1$, a member of G_t at t solves

$$\max u(c_{1t}, c_{2t+1}) \tag{12}$$

with the constraints

$$c_{1t} = e_1 - s_t - q_t m_t, \tag{13}$$

and

$$c_{2t+1} = e_2 + R_t s_t + q_{t+1} m_t, \tag{14}$$

and $(c_{1t}, c_{2t+1}) \geq 0$.

RCE with money

A RCE equilibrium is sequence of prices and quantities $\{q_t, R_t, c_{1t}, c_{2t+1}, m_t, s_t\}$ such that $c_{21} = e_2 + q_1 M$; given $\{R_t, q_t\}$, $(c_{1t}, c_{2t+1}, s_t, m_t)$ solves the maximization problem of G_t for all $t \geq 1$; and all market clear

$$c_{1t} + c_{2t} = e_1 + e_2 \tag{15}$$

$$m_t = M \tag{16}$$

Notice by $c_{21} = e_2 + q_1 M$ and $c_{11} = e_1 - q_1 M - s_1$ and the market clearing condition we must have $s_1 = 0$, and repeat these steps, we conclude

$$s_t = 0 \tag{17}$$

we are interested in a monetary equilibrium with $q_t > 0$ for all t.

Conditions for monetary equilibrium

- with q_t, the first order condition requires

$$\mu(e_1 - q_t m_t, e_2 + q_{t+1} m_t) = \frac{q_{t+1}}{q_t} \tag{18}$$

The solution m_t to (18) gives the money demand function as long as $0 < m_t < e_1/q_t$, the first inequality is true by definition in a monetary equilibrium, and the the second we can guarantee by assume

$$\mu(c_1, c_2) \to \infty \text{ as } c_1 \to 0 \tag{19}$$

- in equilibrium $m_t = M$, so (18) define a dynamic relationship between q_t and q_{t+1}

$$q_{t+1} = f(q_t) \tag{20}$$

Conditions for monetary equilibrium (continued)

- $f(0) = 0$, this is easy

$$q_{t+1} = q_t \mu(e_1 - q_t m_t, e_2 + q_{t+1} m_t) \tag{21}$$

- $f'(0) = \mu(e_1, e_2)$, to see this we have

$$f'(q_t) = \frac{\partial q_{t+1}}{\partial q_t} \tag{22}$$

  differentiating (21) we have

$$dq_{t+1} = dq_t \mu(e_1 - q_t m_t, e_2 + q_{t+1} m_t) - q_t d\mu_t \tag{23}$$

  and evaluating it at $(0, 0)$ we then have

$$f'(0) = \mu(e_1, e_2) \tag{24}$$

- q is a steady-state value such that

$$f(q) = q \tag{25}$$

Conditions for monetary equilibrium (continued)

Lemma

$f'(0) \geq 1$ *implies there is no solution to* $f(q) = q$ *while* $f'(0) = \mu(e_1, e_2) < 1$ *implies that there is exactaly one solution.*

Conditions for monetary equilibrium (continued)

Proof.

Note that solutions to $f(q) = q$ satisfy $T(q) = 0$, where

$$T(q) = -u_1(e_1 - qM, e_2 + qM) + u_2(e_1 - qM, e_2 + qM).$$

and

$$T'(q) = M(u_{11} + u_{22}) < 0 \quad (26)$$

since $T'(q) < 0$, there cannot be more than one solution to $T(q) = 0$, or $f(q) = q$. Since $\lim_{q \to \frac{e_1}{M}} T(q) = -\infty$ then there exists a solution if and only if

$$T(0) > 0, \quad (27)$$

which holds if and only $\mu(e_1, e_2) < 1$. Notice this is exactly the condition for the nonmonetary equilibrium being inefficient. □

Uniquess of the monetary equilibrium

- The steady-state equilibrium is unique if $f'(0) < 1$. Denote $q_t = q^*$ as the particular equilibrium. Notice in $q_t = q^*$, we have

$$\mu(e_1 - q^* M, e_2 + q^* M) = 1$$

An Example

Consider an example with the log-linear utility function, $u(c_{1t}, c_{2t+1}) = \log(c_{1t}) + \log(c_{2t+1})$. This allows us to solve (18) explicitly for the money demand function,

$$m_t = m(q_t, q_{t+1}) = \frac{e_1 q_{t+1} - e_2 q_t}{q_t q_{t+1}} \tag{28}$$

which satisfies $m_t > 0$ if $q_{t+1}/q_t > e_2/e_1$. and $m_t = 0$ if $q_{t+1}/q_t \leq e_2/e_1$. Consider an special case, $e_2 = 0$ and

$$q_t = \frac{e_1}{M} = q^* \tag{29}$$

so in this case, there is a unique monetary equilibrium.

An Example

For more general case, $e_1 > 0$, $e_2 > 0$. We have

$$q_{t+1} = f(q_t) = \frac{e_2 q_t}{e_1 - M q_t} \tag{30}$$

in this case, $f'(q) > 0, f''(q) > 0$ and $f(q) \to \infty$ as $q \to \frac{e_1}{M}$. As always if $f'(0) = \mu = \frac{e_2}{e_1} < 1$ then there is a unique monetary steady-state with

$$q_t = \frac{e_1 - e_2}{M} = q^* \tag{31}$$

but any $q_1 \in [0, q^*]$ is also equilibrium.

Uniquess of the monetary equilibrium