Notes on the Ramsey Model

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Introduction: Ramsey-Koopman-Case model

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- The Calculus of Variations
- Phase diagram

The Calculus of Variations: Statement of the problem

- Consider the following intergral:

$$J = \int_a^b f(t, x(t), \dot{x}(t))dt$$

where a and b are some constants. The function $x(t)$can either be real-valued or Rn valued. Let X be the set of all differentiable functions defined on the closed interval [a, b] .

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- **Problem:**Choose a function $x(t) \in X$ such that J is maximized (minimized) subject to the terminal conditions

$$x(a) = \alpha, x(b) = \beta.$$

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Consider the following example: Find the curve which joints two points on the plane with the minimum distance. A curve joining A and B can be represented by $x(t)$ with $x(a) = \alpha$, and $x(b) = \beta$.

- The distance: the distance along each infinitesimal segment of x(t) is

$$ds = \sqrt{(dt)^2 + (dx)^2} = \sqrt{1 + (\frac{dx}{dt})^2} dt = \sqrt{1 + [\dot{x}(t)]^2} dt \qquad (1)$$

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- The problem is then to find a function $x(t) \in X$ to minimize the above integral subject to $x(t)$ with $x(a) = \alpha$, and $x(b) = \beta$.

The Calculus of Variations: Example 2

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- the problem is find a consumption path to maximize J

Euler Equation

- Let X be the set of all real-valued (and single valued) continuous differentiable functions defined on the closed interval [a, b]. We want to find a function $x(t)$ in X which maximizes (or minimizes) the following integral:

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- subject to $x(a) = \alpha$ and $x(b) = \beta$. We assume that f possesses continuous first and second partial derivatives with respect to all its arguments.Suppose that there exist a function

$$\hat{x}(t) \in X \tag{6}$$

that maximizes (or minimizes) J.

Euler Equation (continued)

- Consider an arbitrary differentiable function $h(t) \in X$, such that $h(a) = 0$ and $h(b) = 0$. Let ε be a real number and define

$$x_\varepsilon(t) = \hat{x}(t) + \varepsilon h(t). \tag{7}$$

Notice $x_\varepsilon(a) = \alpha$ and $x_\varepsilon(b) = \beta$, so $x_\varepsilon(t)$ satisfies the constraint.

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- By assumption $J[x_\varepsilon]$ attains its maximum (minimum) when $\varepsilon = 0$. so we have

$$\frac{\partial J[x_\varepsilon]}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0 \tag{8}$$

Euler Equation (continued)

- But we have

$$\frac{\partial J[x_\varepsilon]}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon}\int_a^b f[t, \hat{x}(t) + \varepsilon h(t), \hat{x}'(t) + \varepsilon \dot{h}(t)]dt$$
$$= \int_a^b f_x' h(t)dt + \int_a^b f_{\dot{x}}' \dot{h}(t)dt \qquad (9)$$

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$$= \int_a^b f'_x h(t) dt + \int_a^b f'_{\dot{x}} \dot{h}(t) dt \qquad (9)$$

- Integration by parts yields:

$$\int_a^b f'_{\dot{x}} \dot{h}(t) dt = f'_{\dot{x}} \cdot h|_a^b - \int_a^b \left[\frac{\partial f'_{\dot{x}}}{dt} h(t) \right] dt \qquad (10)$$
$$= -\int_a^b \left[\frac{\partial f'_{\dot{x}}}{dt} h(t) \right] dt, \text{ as } h(a) = h(b) = 0$$

Euler Equation (continued)

- But we have:

$$\frac{\partial J[x_\varepsilon]}{\partial \varepsilon}\Big|_{\varepsilon=0} = \int_a^b \left[f'_x - \frac{\partial f'_{\dot{x}}}{dt} \right] h(t)dt = 0 \tag{11}$$

since this true for any $h(t) \in X$, with $h(a) = h(b) = 0$. So we conclude that

$$f'_x - \frac{\partial f'_{\dot{x}}}{dt} = 0 \text{ for all } t \in [a, b] \tag{12}$$

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- Therefore, we have the following necessary condition for the maximum (minimum) of the integral J

$$\frac{\partial f(t, x(t), \dot{x}(t))}{\partial x(t)} = \frac{d}{dt}\left[\frac{\partial f(t, x(t), \dot{x}(t))}{\partial \dot{x}(t)}\right] \tag{13}$$

This equation is called **Euler's equation** (condition).

Application (continued)

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- so we have

$$\frac{\dot{x}(t)}{\sqrt{1 + [\dot{x}(t)]^2}} = const$$

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- since $x(a) = \alpha, x(b) = \beta$, we obtain

$$x(t) = (\frac{\alpha - \beta}{a - b})t + \frac{a\beta - \alpha b}{a - b} \tag{18}$$

Back to the Ramsey Model-Assumptions

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- Firms: Each firm has access to the production technology

$$Y = F(K, AL) \tag{19}$$

F has same properties as in chapter 1. The profit function of the representative firm in each period is given by

$$\Pi_t = Y_t - (r_t + \delta)K_t - w_t L_t \tag{20}$$

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- There are H identical households, where H is a fixed number. Each household is composed of M_t identical members. Each member supplies one unit of labor at every point in time. The size of each household grows at rate n:

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- Households maximizes

$$U = \int_{t=0}^{\infty} e^{-\rho t} U(C_t) \frac{L_t}{H} dt \tag{22}$$

where C is the consumption of a member, $u(C)$ is the corresponding utility level, $\frac{L_t}{H}$ is the number of members of the household, and ρ is the discount rate for the future.

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- Assume $H = 1$ as a normalization.

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$$\frac{U'(C_t)}{U'(C_{t+j})} = \frac{P_t}{P_{t+j}} \tag{25}$$

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$$-\frac{\partial\left(\frac{C_t}{C_{t+j}}\right)/\left(\frac{C_t}{C_{t+j}}\right)}{\partial\left(\frac{P_t}{P_{t+j}}\right)/\frac{P_t}{P_{t+j}}} = \frac{1}{\theta} \tag{27}$$

Firm's behavior

- firm profit maximization

$$\frac{\partial \Pi}{\partial K} = F'_K - (r + \delta) = 0 \tag{28}$$

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- Hence there is zero profit.

Firm's behavior (continued)

- Again $f(k) = F(\frac{K}{AL}, 1)$, so we have $F'_K = r + \delta = f('k)$. The proof is easy

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- we can rewrite it as

$$w = A[f(k) - f'(k)k]$$

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- **budget constraint**: Suppose the representative household's initial capital holdings is K_0 and it is K_s at time $t = s$.

$$e^{-R_s} K_s + \int_0^s e^{-R_t} C_t L_t dt = K_0 + \int_0^s e^{-R_t} w_t L_t dt \tag{37}$$

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- differentiating both sides w.r.t. s

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- this holds for any time s, so we have

$$C_t L_t + \dot{K}_t = r_t K_t + w_t L_t \tag{39}$$

where the LHS is the total expenditure, while the RHS is the total income.

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- **The no-Ponzi-game condition:** taking limit of the life-time budget constraint to the limit we have

$$\lim_{s \to \infty} e^{-R_s} K_s + \lim_{s \to \infty} \int_0^s e^{-R_t} C_t L_t dt = K_0 + \lim_{s \to \infty} \int_0^s e^{-R_t} w_t L_t dt \tag{40}$$

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it is to say

$$\int_0^\infty e^{-R_t} C_t L_t dt \leq K_0 + \lim_{s\to\infty} \int_0^\infty e^{-R_t} w_t L_t dt \tag{44}$$

the life time present value of consumption can not exceed the life-time income plus initial wealth.

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$$c_t + \dot{k}_t = f(k_t) - (g + n + \delta) k_t \tag{47}$$

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- where $c_t = \frac{C_t}{A_t}$

Transformation

- Transformation of utility function to per-effective labor economy

$$U = \int_t^{\infty} e^{-\rho t} \frac{A_t^{1-\theta} c_t^{1-\theta}}{1-\theta} L_t dt = A_0^{1-\theta} L_0 \int_t^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt, \quad (48)$$

where $\beta = \rho - (1-\theta)g - n$. We assume $\beta > 0$. Without loss of generality, we assume assume $A_0 = L_0 = 1$.

Transformation (continued)

- Transformation of budget constraint

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- can be written as

$$A_0 L_0 \int_0^\infty e_t^{-R_t} e^{(g+n)t} c_t dt \leq K_0 + A_0 L_0 \int_0^\infty e^{-R_t} e^{(g+n)t} \omega_t dt \tag{50}$$

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- where $\omega_t = \frac{w_t}{A_t}$, or

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Transformation (continued)

- Transformation of budget constraint

$$\int_0^\infty e^{-R_t} C_t L_t dt \leq K_0 + \int_0^\infty e^{-R_t} w_t L_t dt \tag{49}$$

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- No-Ponzi-game condition:

$$\lim_{s\to\infty} e^{-R_s} K_s = A_0 L_0 \lim_{s\to\infty} e^{-R_s} e^{(g+n)s} k_s = 0 \tag{52}$$

Maximization (continued)

- We can set the constrained maximization problem using the Lagrange method:

$$\max_{c_t} \int_{t=0}^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt$$
$$+\lambda[k_0 + \int_0^{\infty} e^{-R_t} e^{(g+n)t} \omega_t dt - \int_0^{\infty} e^{-R_t} e^{(g+n)t} c_t dt] \quad (53)$$

where $\lambda > 0$ is the Lagarangian multiplier.

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- Equation (57) is known as the Euler equation for this maximization problem.

Solution of a Planer

- The objective function

$$\max \int_{t=0}^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt \tag{58}$$

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- Define

$$J = \int_{t=0}^{\infty} e^{-\beta t} \left\{ \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left[f(k_t) - (g+\delta+n)k_t - c_t - \dot{k}_t \right] \right\} dt \tag{60}$$

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- Notice since neither $\dot{c}_t$,$\dot{\lambda}_t$ appear. So Euler equation w.r.t c_t and λ_t is easy

$$c_t^{-\theta} = \lambda_t \tag{62}$$

$$f(k_t) - (g+\delta+n)k_t - c_t = \dot{k}_t \tag{63}$$

Solution of a Planer

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$$\frac{\partial f(t, x(t), \dot{x}(t))}{\partial x(t)} = \frac{d}{dt}[\frac{\partial f(t, x(t), \dot{x}(t))}{\partial \dot{x}(t)}] \tag{64}$$

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- or

$$-\frac{\dot{\lambda}_t}{\lambda_t} = f'(k_t) - \delta - \beta - g - n \tag{67}$$

The Dynamics of the economy

- **The Phase Diagram:** The dynamics of c is described by

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[f'(k) - \delta - \beta - g - n] \tag{68}$$

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- **The Phase Diagram:** The dynamics of c is described by

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[f'(k) - \delta - \beta - g - n] \tag{68}$$

- and the capital follows

$$\dot{k}_t = f(k_t) - (\delta + g + n)k_t - c_t$$

Steady-state

- $\dot{c} = 0$ implies

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- or
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- and conversly, if $k_t < \bar{k}$, we have $\frac{\dot{c}_t}{c_t} > 0$, or consumption would increases.

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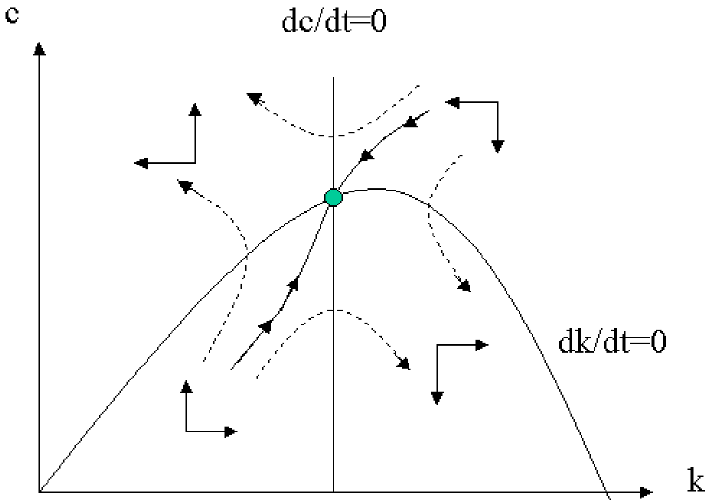
- which is hump-shaped in the $c - k$ space. Above the curve, consumption is too hight, hence $\dot{k}_t < 0$ or $k_t \downarrow$; below this curve, consumption is too low, hence $\dot{k}_t > 0$ or $k_t \uparrow$.

Phase Diagram



Phase Diagram (continued)

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- The solution to this two-variable differential equation system is essentially the policy $c_t = c(k_t)$ for arbitrary t. Hence, given k_0, we need only to know $c_0 = c(k_0)$then the Euler equation and the instantaneous budget constraint will guide the economy towards the steady state along the saddle path.

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- However, the two differential equations (the Euler equation and the instantaneous budget constraint) are not sufficient to give the optimal policy rule unless some extra conditions are given [differential equations give only the optimal rate of change, not the levels.

The Golden rule and balanced growth path

- The S-S satisfies the modified golden rule (= Golden rule if $\beta = 0$):

$$f^{'}(k) = \beta + \delta + g + n.$$

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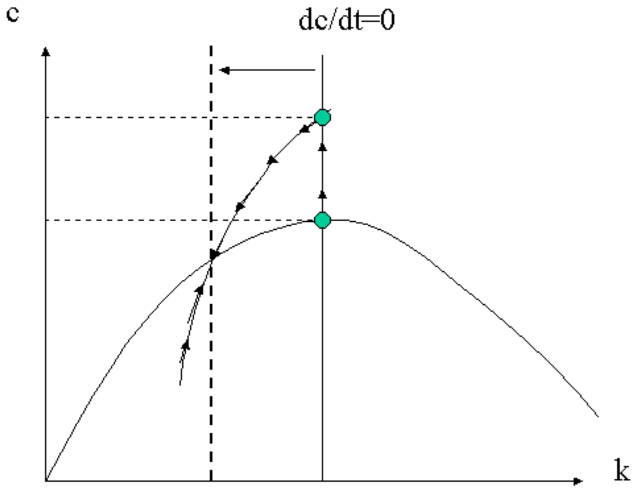
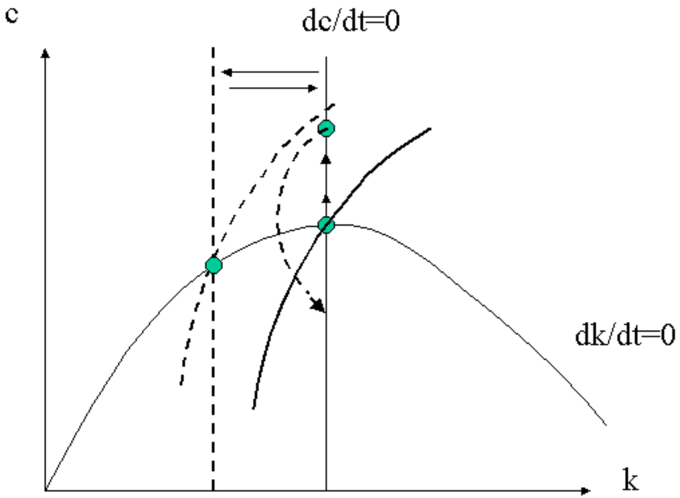
- and by definition $\alpha = \frac{f'(k)k}{f(k)}$

$$1 - \frac{c}{y} = (g+n+\delta)\frac{\alpha}{f'(k)} = \alpha(\frac{g+n+\delta}{\beta+g+n+\delta}) < \alpha.$$

Unexpected & Permanent Changes in discount rate



Unexpected & Transitory Changes in discount rate



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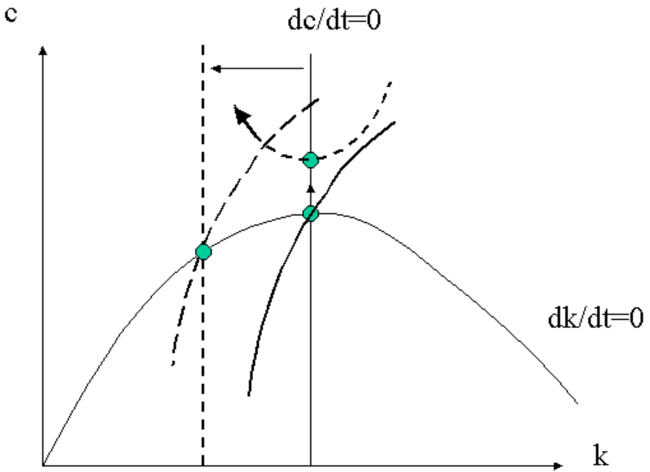

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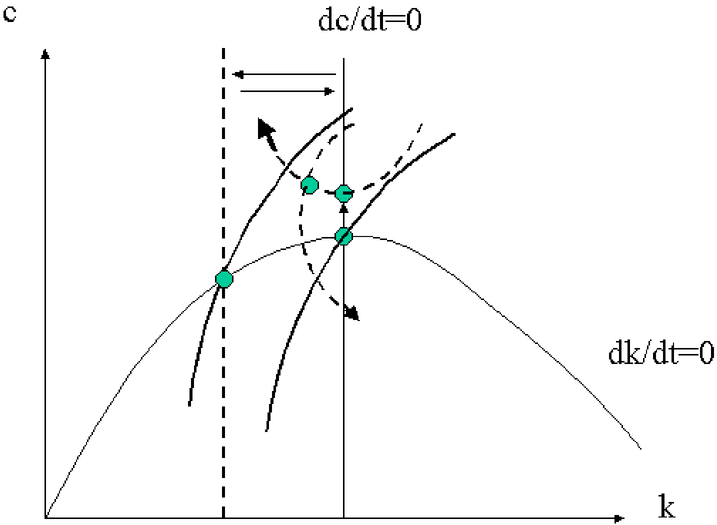

Algebraic Analysis

- Linearization around the S-S. we have

$$\dot{c}_t = c_t \frac{1}{\theta}(f'(k_t) - \beta - \delta - g - n) \tag{73}$$

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- notice that $\frac{\partial \dot{c}_t}{\partial c} = 0$ for $c = \bar{c}$ and $\frac{\partial \dot{c}_t}{\partial k} = \frac{\bar{c}}{\theta} f''(\bar{k})$, so we have

$$\begin{aligned} \dot{c}_t &\simeq 0 + \frac{\bar{c}}{\theta} f''(\bar{k})(k_t - \bar{k}) \\ &\equiv -\tau(k_t - \bar{k}) \end{aligned} \tag{75}$$

Algebraic Analysis (continued)

- The capital equation

$$\dot{k}_t = f(k_t) - (\delta + g + n)k_t - c_t \tag{76}$$

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$$\frac{\partial \dot{k}_t}{\partial c} = -1; \frac{\partial \dot{k}_t}{\partial k} = f'(k) - (g + n + \delta) = \beta \tag{78}$$

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- so we have

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Solving the difference equation

- We have obtain

$$\dot{c}_t = -\tau(k_t - \bar{k}) \quad (80)$$

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$$\{\mu_1, \mu_2\} = \frac{1}{2}(\beta \pm \sqrt{\beta^2 + 4\tau}) \tag{85}$$
- clearly one of them is positive and the other negative. Let $\mu_1 < 0$ and $\mu_2 > 0$

Solving the difference equation

- Let $\mu_1 < 0$ and $\mu_2 > 0$, we must have

$$k_t - \bar{k} = a_1 e^{\mu_1 t} + a_2 e^{\mu_2 t}, \tag{86}$$

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- or

$$a_2 = \frac{1}{\mu_1 - \mu_2}[c_0 - \bar{c} - \mu_2(k_0 - \bar{k})] \tag{92}$$

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- and $a_2 = 0$, so we by $k_0 = \bar{k} + a_1 + a_2$, we have

$$a_1 = k_0 - \bar{k} \tag{96}$$

Solving two coefficients

- we already have

$$c_0 - \bar{c} = \mu_2(k_0 - \bar{k}) \tag{95}$$

- and $a_2 = 0$, so we by $k_0 = \bar{k} + a_1 + a_2$, we have

$$a_1 = k_0 - \bar{k} \tag{96}$$

- hence the solutions are

$$k_t - \bar{k} = (k_0 - \bar{k})e^{\mu_1 t} \tag{97}$$

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Anticipated permanent change in

β

Given an anticipated permanent change in [redacted] that will take place later in period $t = \tau \geq 0$, what's the optimal response of consumption now in period t = 0?

- The system is characterized by

$$k_t - \bar{k} = a_1 e^{\mu_1 t} + a_2 e^{\mu_2 t}, \tag{99}$$

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- where

$$a_1 = \frac{1}{\mu_1 - \mu_2}[\mu_1(k_0 - \bar{k}) -] \tag{101}$$

$$a_2 = \frac{1}{\mu_1 - \mu_2}[(c_0 - \bar{c}) - \mu_2(k_0 - \bar{k})] \tag{102}$$

Anticipated permanent change in

β

- Starting from the steady-state, $k_0 = \bar{k}$ and suppose the optimal consumption is

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Anticipated permanent change in β

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- then we have

$$a_1 = -\frac{1}{\mu_1 - \mu_2}\Delta c \tag{104}$$

and

$$a_2 = \frac{1}{\mu_1 - \mu_2}\Delta c \tag{105}$$

The path of consumption and capital

- τ period later, we have

$$k_\tau - \bar{k} = \frac{\Delta c}{\mu_1 - \mu_2}[-e^{-\mu_1 t} + e^{\mu_2 t}] \tag{106}$$

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- at $t = \tau$, a permanent change in β, hence the steady-state changes from $\{\bar{k}, \bar{c}\}$ to $\{k^*, c^*\}$

The path of consumption and capital

- so we must have

$$c_\tau - k^* = \mu_2(k_\tau - k^*) \tag{108}$$

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$$\begin{aligned} & \bar{c} - c^* + \frac{\Delta c}{\mu_1 - \mu_2}[-\mu_2 e^{-\mu_1 t} + \mu_1 e^{\mu_2 t}] \\ = \; & \mu_2 [\bar{k} - k^* + \frac{\Delta c}{\mu_1 - \mu_2}(-e^{\mu_1 \tau} + e^{\mu_2 \tau})] \end{aligned} \tag{109}$$

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- or

$$\frac{\Delta c}{\mu_1 - \mu_2}(\mu_1 - \mu_2)e^{\mu_2 \tau} = [\mu_2(\bar{k} - k^*) - (\bar{c} - c^*)] \tag{110}$$

The path of consumption and capital

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$$\bar{c} - c^* = \beta(\bar{k} - k^*) \tag{113}$$

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- have in the new and old steady-state

$$\bar{c} - c^* = \beta(\bar{k} - k^*) \tag{113}$$

- or

$$\Delta c = (\mu_2 - \beta)(\bar{k} - k^*)e^{-\mu_2 \tau} > 0 \tag{114}$$