Notes on the Solow Model

Pengfei Wang

Hong Kong University of Science and Technology

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Introduction: Basic Facts about Economic Growth

- Small difference in growth rate can make large difference in income in the long run.

Real GDP per Capita (US$, 1985 Price)

Country	1870	1989	$\bar{g}$	2003 (Projection)
Australia	3,123 (5×Japan)	13,584	1.2%	16,053 (0.73×Japan)
Japan	618	15,101	2.7%	21,928 (35×618)
United Sttates	2,247	18,317	1.8%	23,514

Introduction: Basic Facts about Economic Growth (continued)

Real GDP per Capita (US$, 2000 price)

Country	Year 2000	g	Year 2078
India	500	5%	22,477 (10,655 if $g = 4\%$)
China	728	7%	142,582
US	30,000 (40×China)	2%	140,583

Questions to be Addressed

- How important is the rate of capital accumulation to a nation's economic growth?
- What are the economic forces that ultimately allow poor countries to catch up with the richest countries in living standards?
- How does a nation's growth rate evolve over time? Is there a limit? If so, what determines the limit — the rate of saving? the rate of population growth? or the amount of natural resources?

Assumption

- There is only a single good, so relative prices do not play any role;
- There is no money and no government spending, so government does not play any role;
- Full employment at all time, so unemployment does not matter;
- Except inputs, the production technology does not change over time;
- There are only two types of inputs: capital and labor;
- The rates of saving, depreciation, population growth, and technology progress are constant.

These features are both defects and virtues of the model.

Assumption (continued)

More specifically and mathematically, we have

- Production function

$$Y = F(K_t, A_t L_t) \tag{1}$$

where K denotes capital stock, A denotes labor productivity, L denotes labor units.

Properties:

$$\begin{aligned} F_1^{'} &> 0, F_2^{'} > 0, F_{12}^{''} > 0, F_{21}^{''} > 0, F_1^{''} < 0, F_2^{''} < 0 \quad (2) \\ F(\lambda K, A\lambda L) &= \lambda F(K, L) \;\; \textit{for } \lambda > 0 \end{aligned}$$

$$\begin{aligned} lim_{x \to 0} F'(x, y) &= \infty; lim_{y \to 0} F'(x, y) = \infty \quad (3) \\ lim_{x \to \infty} F'(x, y) &= 0; lim_{y \to \infty} F'(x, y) = 0 \end{aligned}$$

Assumption (continued)

More specifically and mathematically, we have

- Resource Allocation

$$\begin{aligned} S_t &= I_t = sY_t \\ C_t + I_t &= Y_t \end{aligned} \quad (4)$$

- Evolution of Inputs

$$\begin{aligned} \dot{K}_t &= I_t - \delta K_t \\ \dot{A}_t &= gA_t \\ \dot{L}_t &= nL_t \end{aligned} \quad (5)$$

Model Dynamics

- Per-effective labor economy: Define $x = \frac{X}{AL}$ then

$$y = \frac{F(K, AL)}{AL} = F(\frac{K}{AL}, 1) = f(k) \tag{6}$$

$$c + i = y \tag{7}$$

$$\dot{k} + (g+n)k = i - \delta k \tag{8}$$

where

$$\dot{k} = \frac{d(\frac{K}{AL})}{dt} = \frac{\dot{K}}{AL} - \frac{K(\dot{A}L + A\dot{L})}{(AL)^2} = \frac{\dot{K}}{AL} - (g+n)k \tag{9}$$

and

$$\frac{\dot{K}}{AL} = i - \delta k \tag{10}$$

Model Dynamics (continued)

the property of $f(k) = F(\frac{K}{AL}, 1)$

$$f'(k) = F_1^{'}(K, AL) \tag{11}$$

$$f'(k) = \frac{\partial \left[\frac{F(K,AL)}{AL} \right]}{\partial(\frac{K}{AL})} = \frac{\partial F(K, AL)}{\partial K} = F_1(K, AL) > 0 \tag{12}$$

$$f^{''}(k) = \frac{\partial F_1(K, AL)}{\partial(\frac{K}{AL})} = ALF_{11}(K, AL) < 0 \tag{13}$$

Steady-State

A steady state of a dynamic system is a situation where all the variables in the system are constant, $x_t = \bar{x}$. Assuming that a steady state exists in the transformed Solow model, then

$$\dot{k} = \dot{c} = \dot{y} = \dot{i} = 0, \tag{14}$$

i.e

$$\frac{X}{AL} = const \text{ for } X = K, Y, C, I \tag{15}$$

so

$$\frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{\dot{I}}{I} = g + n \tag{16}$$

this is so called balanced growth path.

Stabability of Steady-State

In S-S, (8) implies

$$\bar{\imath} = (\delta + g + n)\bar{k} \tag{17}$$

and

$$\dot{k}_t = sf(k_t) - (g + n + \delta)k_t = \pi(k_t) \tag{18}$$

the steady-state $\bar{k}$ is given by

$$sf(\bar{k}) = (g + n + \delta)\bar{k} \tag{19}$$

Notice

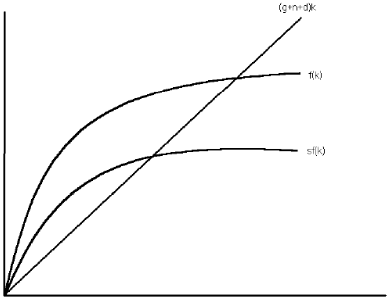
$$\pi'(k) = sf'(k) - (g + n + \delta); \pi''(k) = sf''(k) < 0 \tag{20}$$

so we have

$$\dot{k}_t > 0 \text{ if } k_t < \bar{k} \ ; \dot{k}_t < 0 \text{ if } k_t > \bar{k} \tag{21}$$

Stabability of Steady-State

The phase diagram



Effects of Saving rate

In S-S we have

$$sf(\bar{k}) = (g + n + \delta)\bar{k}. \tag{22}$$

totally differentiating this equation gives

$$f(k)ds + sf'(k)dk = (g + n + \delta)dk, \tag{23}$$

which implies

$$\frac{dk}{ds} = \frac{f(k)}{(g + n + \delta) - sf('k)} = \frac{f(k)}{s\frac{f(k)}{k} - sf'(k)} \tag{24}$$

or

$$\frac{dk/k}{ds/s} = \frac{f(k)}{f(k) - f'(k)k} = \frac{1}{1-\alpha} > 0 \tag{25}$$

Effects of Saving rate (continued)

Note 1: α is output elasticity of capital, we must have

$$0 < \alpha < 1 \tag{26}$$

Proof:

$$\alpha = \frac{f'(k)k}{f(k)} > 0 \tag{27}$$

is easy to see.

Effects of Saving rate (continued)

To see $\alpha < 1$, we have

$$\alpha = \frac{F_1'(K, AL)K}{F(K, AL)} \tag{28}$$

Notice by constant return to scale, we have

$$F(\lambda K, \lambda AL) = \lambda F(K, AL) \tag{29}$$

totally differentiating with respect to λ, and evaluated $\lambda = 1$

$$F_1'(K, AL)K + F_2'(K, AL)AL = Y \tag{30}$$

and $F_2' > 0$, so we have

$$\alpha < 1 \tag{31}$$

Effects of Saving rate (continued)

- Effect on output

$$\frac{dy}{ds} = \frac{df}{dk}\frac{dk}{ds} \tag{32}$$

we have

$$\frac{dy/y}{ds/s} = (\frac{df}{dk}\frac{k}{y})(\frac{dk}{ds}\frac{s}{k}) = \frac{\alpha}{1-\alpha} > 0$$

Effects of Saving rate (continued)

- Effect on consumption

$$c = (1-s)f(k) \tag{33}$$

we have

$$\begin{aligned} \frac{dc}{ds} &= -f(k) + (1-s)f'\frac{dk}{ds} \\ &= -f(k) + (1-s)(\frac{k}{f}f')\frac{f}{k}(\frac{dk}{ds}\frac{s}{k})\frac{k}{s} \\ &= -f(k) + (1-s)\frac{\alpha}{1-\alpha}\frac{f(k)}{s} \end{aligned} \tag{34}$$

so we have

$$\begin{aligned} \frac{dc/c}{ds/s} &= \left[-f(k) + (1-s)\frac{\alpha}{1-\alpha}\frac{f(k)}{s}\right]\frac{s}{(1-s)f(k)} \\ &= -\frac{s}{1-s} + \frac{\alpha}{1-\alpha} \end{aligned} \tag{35}$$

Effects of Saving rate (continued)

- The golden rule: The golden rule is the best S-S where S-S consumption is maximized. Since S-S consumption depends on the saving rate, we can find the golden rule rate of saving such that S-S consumption is maximized:

$$\frac{dc}{ds} = 0 \tag{36}$$

or

$$s = \alpha \tag{37}$$

Growth Effects of Saving rate (continued)

- Growth Effect:

$$\dot{k} = sf(k) - (g + n + \delta)k \tag{38}$$

$$\begin{aligned}
\frac{d\dot{k}}{ds} &= f(k) + sf'(k)\frac{dk}{ds} - (g + n + \delta)\frac{dk}{ds} \\
&= f(k) + [f'(k)k - \frac{(g + n + \delta)k}{s}]\frac{dk}{ds}\frac{s}{k} \\
&= f(k) + (f'(k)k - f(k))\frac{1}{1 - \alpha} \\
&= f(k) + f(k)\frac{\alpha - 1}{1 - \alpha} \\
&= 0
\end{aligned} \tag{39}$$

Transitional Dynamics

- Algebraic analysis:

$$\begin{aligned} \dot{k}_t &= sf(k_t) - (g+n+\delta)k_t \\ &\simeq (sf'(\bar{k}) - (g+n+\delta))(k_t - \bar{k}) \end{aligned} \tag{40}$$

since $sf'(\bar{k}) = sf'(\bar{k})\frac{\bar{k}}{\bar{f}}\frac{\bar{f}}{\bar{k}} = \alpha\frac{s\bar{f}}{\bar{k}} = \alpha(g+n+\delta)$, we have

$$\begin{aligned} \dot{k}_t &\simeq -(1-\alpha)(g+n+\delta)(k_t - \bar{k}) \\ &\equiv -\lambda(k_t - \bar{k}) \end{aligned} \tag{41}$$

this says that near S-S, the growth rate is approximately constant and equal to $-\lambda$. Hence $\lambda = (1-\alpha)(g+n+\delta)$ is called the speed of convergence.

- solving this equation implies

$$\ln(k_t - \bar{k}) + const = -\lambda t \tag{42}$$

notice that k_0 is given so we have

$$\ln(k_0 - \bar{k}) + const = 0 \tag{43}$$

Transitional Dynamics

- solving this equation implies

$$\ln(k_t - \bar{k}) + const = -\lambda t \tag{46}$$

notice that k_0 is given so we have

$$\ln(k_0 - \bar{k}) + const = 0 \tag{47}$$

hence

$$\ln(k_t - \bar{k}) = \ln(k_0 - \bar{k}) - \lambda t \tag{48}$$

or

$$k_t - \bar{k} = (k_0 - \bar{k})e^{-\lambda t} \tag{49}$$

Transitional Dynamics (continued)

- Now consider a small change of s on k_t

$$\frac{dk_t}{ds} = \frac{d\bar{k}}{ds}(1 - e^{-\lambda t}) \geq 0 \tag{50}$$

which is zero at $t = 0$ and increases in a diminishing manner towards $\frac{d\bar{k}}{ds}$ with t.

- growth effect on $\dot{k}_t$, by $\dot{k}_t = -\lambda(k_t - \bar{k})$ we have

$$\begin{aligned} \frac{d\dot{k}_t}{ds} &\simeq -\lambda(\frac{dk_t}{ds} - \frac{d\bar{k}}{ds}) \\ &\equiv \lambda e^{-\lambda t}\frac{d\bar{k}}{ds} \end{aligned} \tag{51}$$

which is positive at $t = 0$ and decreases exponentially towards zero with t.

Transitional Dynamics (continued)

- The effect on output

$$y_t = f(k_t)$$

hence

$$y_t \simeq f(\bar{k}) + f'(\bar{k})(k_t - \bar{k}) \tag{52}$$

so we have

$$\begin{aligned} \dot{y}_t &\simeq f'(\bar{k})\dot{k}_t \\ &\equiv -\lambda f'(\bar{k})(k_t - \bar{k}) \\ &\simeq -\lambda(y_t - \bar{y}) \end{aligned} \tag{53}$$

Transitional Dynamics (continued)

- Since we have

$$\dot{y}_t \simeq -\lambda(y_t - \bar{y}) \tag{54}$$

so we can solve

$$y_t - \bar{y} = (y_0 - \bar{y})e^{-\lambda t}. \tag{55}$$

so we have

$$\frac{dy_t}{ds} = (1 - e^{-\lambda t})\frac{d\bar{y}}{ds}, \tag{56}$$

which is similar to the analysis to capital.

- The effect on consumption

$$c_t = y_t - (g+n+\delta)k_t - \dot{k}_t \tag{57}$$

so we have

$$\begin{aligned} \frac{dc_t}{ds} &= \frac{dy_t}{ds} - (g+n+\delta)\frac{dk_t}{ds} - \frac{d\dot{k}_t}{ds} \\ &= (1-e^{-\lambda t})\frac{d\bar{y}}{ds} - (g+n+\delta)\frac{d\bar{k}}{ds}(1-e^{-\lambda t}) - \lambda e^{-\lambda t}\frac{d\bar{k}}{ds} \end{aligned} \tag{58}$$

which is always negative at $t = 0$.

A micro-foundation of aggregate production function

- firms: Suppose there is a continuum of firms measured by $i \in [0, 1]$. Its production function is

$$y(i) = \varepsilon(i) \min [k(i), An(i)] \tag{59}$$

- the total output is

$$Y = \int_0^1 y(i) di \tag{60}$$

- Where $k(i)$ is the capital used by firm i and $n(i)$ is the labor input in such a firm, the aggregate technology level is A.
- firm specific technology level $\varepsilon(i)$ drawn from a common distribution function. $\Pr [\varepsilon(i) \geq \varepsilon] = \varepsilon^{-\sigma}$ with the density function equal to $f(\varepsilon) = \sigma \varepsilon^{-\sigma-1}$.

A micro-foundation of aggregate production function

- timing : Before the realization of $\varepsilon(i)$, firm needs to install capital. After that $\varepsilon(i)$ realizes, and firm i decides whether to produce or not.
- Let r be the interest rate, and w be the real wage and the good's price be unit, the firm's problem can be solved by backward reduction. First consider the labor decision, its profit is

$$\pi(\varepsilon(i)) = \varepsilon(i)An(i) - wn(i) \tag{61}$$

$$s.t\ An(i) \leq k(i) \tag{62}$$

A micro-foundation of aggregate production function

- In this case, the labor decision and output is

$$n(i) = \left\{ \begin{array}{cc} \frac{k(i)}{A} & \text{if } \varepsilon(i) \geq \frac{w}{A} \\ 0 & \text{otherwise} \end{array} \right\} \tag{63}$$

$$y(i) = \left\{ \begin{array}{cc} \varepsilon(i)k(i) & \text{if } \varepsilon(i) \geq \frac{w}{A} \\ 0 & \text{otherwise} \end{array} \right\} \tag{64}$$

A micro-foundation of aggregate production function

- The firm's profit is

$$\pi(i) = \left\{ \begin{array}{cc} \varepsilon(i)k(i) - \frac{k(i)}{A}w & \text{if } \varepsilon(i) \geq \frac{w}{A} \\ 0 & \text{otherwise} \end{array} \right\} \tag{65}$$

- Hence its expected profit when its install capital is :

$$-rk(i) + k(i)\int_{\frac{w}{A}}^{\infty} \left(\varepsilon - \frac{w}{A}\right) f(\varepsilon) d\varepsilon$$

A micro-foundation of aggregate production function

- This implies

$$
\begin{aligned}
r &= \int_{\frac{w}{A}}^{\infty} \left(\varepsilon - \frac{w}{A} \right) f(\varepsilon) d\varepsilon \\
&= \int_{\frac{w}{A}}^{\infty} \varepsilon f(\varepsilon) d\varepsilon - \left(\frac{w}{A} \right)^{1-\sigma} \\
&= \frac{1}{\sigma - 1} \left(\frac{w}{A} \right)^{1-\sigma} \qquad (66)
\end{aligned}
$$

- and the total production:

$$
Y = K \int_{\frac{w}{A}}^{\infty} \varepsilon f(\varepsilon) d\varepsilon = \frac{\sigma}{\sigma - 1} \left(\frac{w}{A} \right)^{1-\sigma} K \qquad (67)
$$

A micro-foundation of aggregate production function

- Interest rate is

$$rK = \frac{1}{\sigma} Y \tag{68}$$

- and the real wage is

$$\left(\frac{w}{A}\right)^{1-\sigma} = \frac{\sigma - 1}{\sigma} \frac{Y}{K}$$

- the total wage payment

$$wN = w\frac{K}{A} \int_{\frac{w}{A}}^{\infty} f(\varepsilon) d\varepsilon = K \left(\frac{w}{A}\right)^{\sigma-1} \tag{69}$$

or we have :

$$wN = \frac{\sigma - 1}{\sigma} Y \tag{70}$$

A micro-foundation of aggregate production function

- Finally use $w = A\left(\frac{\sigma-1}{\sigma}\frac{Y}{K}\right)^{\frac{1}{1-\sigma}}$ to compute the aggregate ouput. we have

$$A\left(\frac{\sigma-1}{\sigma}\frac{Y}{K}\right)^{\frac{1}{1-\sigma}} N = \frac{\sigma-1}{\sigma} Y \tag{71}$$

or we have

$$ANK^{\frac{1}{\sigma-1}}\frac{\sigma-1}{\sigma}^{\frac{\sigma}{1-\sigma}} = Y^{\frac{\sigma}{\sigma-1}} \tag{72}$$

This yields

$$Y = \frac{\sigma}{\sigma-1}K^{\frac{1}{\sigma}}\left[AN\right]^{\frac{\sigma-1}{\sigma}} \tag{73}$$