Notes on Credit Cycles

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Introduction

- The RBC models, financial factors do not matter. Asset pricing are computed as by-product of the model. Technology shocks drive business cycle$\rightarrow$leads to fluctuation in asset price.
- In the economy, durable asset such as land, building, and machinery play a dual role: not only are they factors of production, but these also serve as collateral for loans.
- In an economy with financial friction. Asset price can have a feedback effect on business cycle. Technology shocks drive business cycle$\rightarrow$leads to fluctuation in asset price$\rightarrow$changes in business cycle.
- So technology shocks can be amplified and propagated by asset prices.

Model Setup

- Two agents. Farmer (entrepreneurs) and Gather (worker).
- One input : Land.
- Farmer is credit constrained in equilibrium because Farmer is less patient. The credit limit is constrained by his land value.

- The Farmer maximizes

$$E_t \sum_{\tau=0}^{\infty} \beta^{\tau} x_{t+\tau} \tag{1}$$

$$x_t \geq ck_t \tag{2}$$

$$q_t(k_{t+1} - k_t) + b_t + x_t = (a + c)k_t + \frac{b_{t+1}}{R_t} \tag{3}$$

$$b_{t+1} \leq E_t q_{t+1} k_{t+1} \tag{4}$$

Where $\beta < 1$ is the discount factor, q_t is the land price, x_t is consumption and b_t the the famer's debt.

Worker (Gather)

- The gather (worker) solves a similar problem

$$E_t \sum_{\tau=0}^{\infty} \beta_h^{\tau} x_{ht+\tau} \tag{5}$$

- With the period by period constraint

$$q_t(k_{h,t+1} - k_{h,t}) + b_{h,t} + x_{h,t} = G(k_{h,t}) + \frac{b_{h,t+1}}{R_t} \tag{6}$$

  where x_h the woker's consumption.

- $G(k_{h,t})$ is the total production of the worker. We assume $G(k_{h,t})$ is a strictly increasing and concave function in k_{ht}.

Worker (Gather)

- The first order condition of the worker yields

$$\frac{1}{R_t} = \beta_h \tag{7}$$

And

$$q_t = \beta_h E_t q_{t+1} + \beta_h G'(k_{h,t+1}) \tag{8}$$

So we have

$$R = \frac{1}{\beta} \tag{9}$$

- Define the user's cost of land is

$$u_t = q_t - \frac{E_t q_{t+1}}{R}$$

Worker (Gather)

- So we can write the land price in term future user's cost

$$q_t = \sum R^{-\tau} E_t u_{t+\tau} \tag{10}$$

This implies that the price of land is simply the discounted future user cost or the discounted future marginal productivity of land.

The Farmer (Firm/Entreprenuer)

- We now solve the Farmer's problem. We have two additional assumption
- **Assumption 1**:
$$\beta < \beta_h \tag{11}$$
- **Assumption 2**:
$$c > (\frac{1}{\beta} - 1)a \tag{12}$$
- The implication of these two assumption will be discussed later.

The Farmer (Firm/Entreprenuer)

- The problem of the entreprenuer. The utility is linear function in consumption and the constraints are linear too.
- Conjecture the value function is linear:

$$\varphi_t k_t - \phi_t b_t = \max\left\{(a+c)k_t + \frac{b_{t+1}}{R} - q_t(k_{t+1} - k_t) - b_t + \beta E_t\left[\varphi_{t+\right.\right. \quad (13)$$

- Write the constraint $x_t \geq ck_t$ as

$$ak_t + \frac{b_{t+1}}{R} - q_t(k_{t+1} - k_t) - b_t \geq 0 \quad (\pi)$$

- The borrowing constraint is the same

$$b_{t+1} \leq E_t q_{t+1} k_{t+1} \quad (\mu)$$

The Farmer (Firm/Entreprenuer)

- First order condtion with respect to b_{t+1} is

$$\frac{1}{R}[1+\pi_t] - \beta E_t \phi_t - \mu_t = 0 \tag{14}$$

where π_t and μ_t are Lagarange multiplier of constraint (π) and (μ) respectively.

- compare terms we ahve

$$\phi_t = 1 + \pi_t \tag{15}$$

- This implies that

$$\mu_t = \beta_h[1+\pi_t] - \beta E_t\left[1+\pi_{t+1}\right] \tag{16}$$

- so in the steady-state we have

$$\mu > 0 \tag{17}$$

The Farmer (Firm/Entreprenuer)

- Assuming small shocks, we then assume $\mu_t > 0$. Since $\mu_t(E_t q_{t+1} k_{t+1} - b_{t+1}) = 0$, we conjecture that

$$E_t q_{t+1} k_{t+1} = b_{t+1} \tag{18}$$

- The first order condition with respect to k_{t+1} implies

$$q_t[1 + \pi_t] = \beta E_t \varphi_{t+1} + \mu_t E_t q_{t+1} \tag{19}$$

- Comparing terms we have

$$\varphi_t = c + (1 + \pi_t)a + (1 + \pi_t)q_t \tag{20}$$

The Farmer (Firm/Entreprenuer)

- substituting φ_{t+1} and μ_t out we have

$$\begin{aligned} q_t[1+\pi_t] &= \beta E_t\left[c + (1+\pi_{t+1})a + (1+\pi_{t+1})q_{t+1}\right] \\ &\quad + \left\{\frac{1}{R}[1+\pi_t] - \beta E_t(1+\pi_{t+1})\right\} E_t q_{t+1} \end{aligned} \tag{21}$$

- or

$$q_t[1+\pi_t] = \beta E_t\left[c + (1+\pi_{t+1})a\right] + \frac{1}{R}[1+\pi_t]E_t q_{t+1} \tag{22}$$

Steady-State

- We want in the steady-state $\pi_t > 0$. We now look at what condition we require
- first we have

$$\left[q - \frac{1}{R}q\right][1+\pi] = \beta\left[c + (1+\pi)a\right] \tag{23}$$

- if $\pi > 0$ we have

$$K = \frac{1}{q - \frac{1}{R}q}\left[(a+q)K - qK\right] \tag{24}$$

- So we have

$$a = q - \frac{1}{R}q = u \tag{25}$$

Steady-State

- since $a = q - \frac{1}{R}q$, we now solve π

$$a(1+\pi)[1-\beta] = \beta c \tag{26}$$

or we have

$$1+\pi = \frac{\beta}{1-\beta}\frac{c}{a} \tag{27}$$

- We require $\pi > 0$ to be consistent with our conjecture

$$\frac{\beta}{1-\beta}\frac{c}{a} > 1 \tag{28}$$

- or we have

$$\frac{c}{a} > \frac{1}{\beta} - 1 \Leftrightarrow c > (\frac{1}{\beta} - 1)a \tag{29}$$

Back to the entrepreneur's problem

- In the steady-state, $\pi > 0$. We now assume it holds in every period. So the constraint

$$ak_t + \frac{b_{t+1}}{R} - q_t(k_{t+1} - k_t) - b_t = 0 \tag{30}$$

  in every period

- We already have

$$E_t q_{t+1} k_{t+1} = b_{t+1} \tag{31}$$

- So the consumption of the entrepreneur is

$$x_t = ck_t \tag{32}$$

Back to the entrepreneur's problem

- In words, the entrepreneur's consumption is simple ck_t in each period. His borrowing constraint is binding in each period. The downpay for each unit of land is $q_t - \frac{E_t q_{t+1}}{R} = u_t$. One unit of land requires q_t payment, among which $\frac{E_t q_{t+1}}{R}$ can borrow from the household use the land as collateral.
- The intuition (Read page 220 on KM). In short
- One additional dollar if buy land can yields additional consumption path

$$0, \frac{c}{u_t}, \frac{a}{u_t}\frac{c}{u_{t+1}}, \frac{a}{u_t}\frac{a}{u_{t+1}}\frac{c}{u_{t+2}}, \ldots \tag{33}$$

- In period t, it can purchase $\frac{1}{u_t}$ lands. It generate $\frac{c}{u_t}$ consumption and $\frac{a}{u_t}$ additional value which can be used to purchase additional $\frac{a}{u_t}\frac{1}{u_{t+1}}$ in period t+1 and generate $\frac{a}{u_t}\frac{c}{u_{t+1}}$ consumption and $\frac{a}{u_t}\frac{a}{u_{t+1}}$ additional value....

Back to the entrepreneur's problem

- We consider one period deviation. Namely the entreprenuer deviate its strategy from equilibrium. There are two other choices
- Choice 1: save this one dollar in bond and invest it in land in the next period, the consumption path is

$$0, 0, R\frac{c}{u_{t+1}}, R\frac{a}{u_{t+1}}\frac{c}{u_{t+2}}, ..., \tag{34}$$

- In period t, there is no consumption, in period t+1, this dollar increases to R, which can purchase $\frac{R}{u_{t+1}}$ unit of land in perio t+1. It then yields $R\frac{c}{u_{t+1}}$ consumption in period t+2 plus $R\frac{a}{u_{t+1}}$ additional value. Again the additional value can purchase $R\frac{a}{u_{t+1}}\frac{1}{u_{t+2}}$ unit land in period t+2 and hence $R\frac{a}{u_{t+1}}\frac{c}{u_{t+2}}$ unit consumption in period t+3, and so on...
- Choice 2, consume it directly, the additional consumption is

$$1, 0, 0, 0, ..., \tag{35}$$

Back to the entrepreneur's problem

- Since in the steady-state, we have $a = u$, we have the first path yield

$$U = \frac{\beta}{1-\beta}\frac{c}{a} \tag{36}$$

- The choice one yields utility

$$U_1 = \frac{\beta^2 R}{1-\beta}\frac{c}{a} = \frac{\beta\beta/\beta_h}{1-\beta} < U \tag{37}$$

- choice 2 yields utility

$$U_2 = 1 \tag{38}$$

- By our assumption $\frac{\beta}{1-\beta}\frac{c}{a} > 1$. So the equilibrium path is consistent with the entrepreneur's optimization problem.
- Notice $U = \frac{\beta}{1-\beta}\frac{c}{a} = 1 + \pi$, explain why $1 + \pi$ is the marginal value of additional dollar to the entrepreneur.

- Is the economy efficient?
- The total maximum output can be obtain by solving

$$\max mG(\frac{1}{m}(\bar{K} - k_t)) + (a + c)k_t \tag{39}$$

- which requires

$$G'(\frac{1}{m}(\bar{K} - k^*)) = a + c \tag{40}$$

- in the equilibrium we have

$$\frac{G'(\frac{\bar{K}-k_t}{m})}{R} = u_t = q_t - \frac{E_t q_{t+1}}{R} \tag{41}$$

Efficience

- or
$$G'(\bar{K}-k_t) = aR = \frac{a}{\beta_h} < \frac{a}{\beta} \tag{42}$$
- hence
$$G'(\frac{\bar{K}-k_t}{m}) - (a+c) < \frac{(1-\beta)a}{\beta} - c < 0 \tag{43}$$
- So there is total ouput loss due credit constraint. Since
$$G'(\frac{\bar{K}-k_t}{m}) < a+c = G'(\frac{1}{m}(\bar{K}-k^*))$$
- There are too much land allocated to the gather and a flow of land from the gather to the farmer(entreprenuer) can increase output.

Dynamics

The rest of equation follows naturally. The equilibrium conditions are characterized by the following 4 equations.

$$K_{t+1} = \frac{1}{u_t}[(a + q_t)K_t - b_t] \tag{44}$$

$$b_{t+1} = E_t q_{t+1} K_{t+1} \tag{45}$$

$$u_t = q_t - \frac{E_t q_{t+1}}{R} \tag{46}$$

$$u_t = q_t - \frac{E_t q_{t+1}}{R} = \frac{1}{R} G' \left[\frac{1}{m} (\bar{K} - K_{t+1}) \right] \tag{47}$$

Log-linearization

- the capital accumulation

$$
\begin{aligned}
\hat{K}_{t+1} &= -\hat{u}_t + \frac{(a+q)}{a}[\hat{K}_t + \frac{q}{a+q}\hat{q}_t + \frac{a}{a+q}\hat{a}_t] - \frac{qK}{aK}\hat{b}_t \\
&= -\hat{u}_t + \frac{(a+q)}{a}\hat{K}_t + \frac{q}{a}\hat{q}_t + \hat{a}_t - \frac{q}{a}\hat{b}_t \\
&= -\hat{u}_t + \frac{2-\beta_h}{1-\beta_h}\hat{K}_t + \frac{1}{1-\beta_h}\hat{q}_t + \hat{a}_t - \frac{1}{1-\beta_h}\hat{b}_t \qquad (48)
\end{aligned}
$$

- The debt condition

$$
\hat{b}_{t+1} = E_t\hat{q}_{t+1} + \hat{K}_{t+1} \qquad (49)
$$

Log-linearization

- And the user's cost

$$\hat{u}_t = -\frac{[\bar{K}-K]\,\frac{1}{m}G''\left[\frac{1}{m}(\bar{K}-K)\right]}{G'\left[\frac{1}{m}(\bar{K}-K)\right]}\frac{K}{\bar{K}-K}\hat{K}_{t+1} \qquad (50)$$

$$= -\frac{1}{\eta}\hat{K}_{t+1} \qquad (51)$$

- so we have

$$\hat{q}_t = \frac{1}{\eta}\frac{R-1}{R}\sum R^{-\tau}\hat{K}_{t+\tau} \qquad (52)$$

- We are interested in a transitory shock $\hat{a}_t = \Delta$ and $\hat{a}_{t+j} = 0$ for all $j > 1$.

Log-linearization

- In period, both capital and debt are state variables, so we have $\hat{K}_t = 0$ and $\hat{b}_t = 0$.
- Using $\hat{u}_t = -\frac{1}{\eta}\hat{K}_{t+1}$, we have

$$(1+\frac{1}{\eta})\hat{K}_{t+1} = \frac{R}{R-1}\hat{q}_t + \Delta \tag{53}$$

- Starting from period $t+1$, $\hat{b}_{t+j} = \hat{q}_{t+j} + \hat{K}_{t+j}$, since there is no further uncertainty,

$$\begin{aligned} \hat{K}_{t+s}(1+\frac{1}{\eta}) &= \frac{2-\beta_h}{1-\beta_h}\hat{K}_{t+s-1} + \frac{1}{1-\beta_h}\hat{q}_{t+s-1} \\ &\quad -\frac{1}{1-\beta_h}\hat{q}_{t+s-1} - \frac{1}{1-\beta_h}K_{t+s-1} \\ &= \hat{K}_{t+s-1} \end{aligned} \tag{54}$$

Log-linearization

- Finally in order to determine $\hat{q}_t$ and $\hat{K}_{t+1}$, using the relationship between the user's cost and capital we then have

$$\hat{q}_t = \frac{1}{\eta}\frac{R-1}{R}\sum R^{-\tau}\hat{K}_{t+\tau} = \frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}\frac{\eta}{1+\eta}}\hat{K}_{t+1} \tag{55}$$

- This solves that

$$\hat{q}_t = \frac{1}{\eta}\Delta \tag{56}$$

and

$$\hat{K}_{t+1} = \frac{1}{1+\frac{1}{\eta}}(1+\frac{R}{R-1}\frac{1}{\eta})\Delta \tag{57}$$

- And for $s \geq 2$.

$$\hat{K}_{t+s} = \frac{1}{(1+\frac{1}{\eta})}\hat{K}_{t+s-1} \tag{58}$$

Log-linearization

- This implies that the model is able to translate i.i.d shock into a persistent movement of capital and asset price.
- Total output in the economy starting from next period

$$Y_{t+s} = G(\bar{K} - K_{t+s}) + (a + c)K_{t+s} \tag{59}$$

- log linearize it yields

$$\hat{Y}_{t+s} = -\frac{G(\bar{K}-K)}{Y}\frac{KG'\left[\bar{K}-K\right]}{G(\bar{K}-K)}\hat{K}_{t+s} \tag{60}$$

$$+\left[1-\frac{G(\bar{K}-K)}{Y}\right](a+c)\hat{K}_{t+s} \tag{61}$$

$$= \theta\hat{K}_{t+s}$$

where $\theta > 0$.

A presentative household maximizes

$$\sum_{t=0}^{\infty} \beta^t E_0[U(C_t, N_t)] \tag{62}$$

with the constraint

$$C_t + \int V_t(i) s_{t+1}(i) di \leq w_t N_t + \int [V_t(i) + D_t(i)] s_t(i) di \tag{63}$$

- This implies

$$V_t(i) = E_t \sum_{j=1}^{\infty} \beta^j \frac{\Lambda_{t+j}}{\Lambda_t} D_{t+j}(i) \tag{64}$$

Firms

- There are a continuum of competitive firms indexed by $i \in [0, 1]$. Firm i's objective is to maximize its discounted dividends,

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} D_t(i), \tag{65}$$

  where D_t represents firm i's dividend in period t and Λ_t the marginal utility of a representative household.

- The production function has constant returns to scale and is given by

$$Y_t(i) = F(K_t(i), A_t N_t(i)), \tag{66}$$

  where A_t represents aggregate labor-augmenting technology that can be either deterministic or stochastic, $N_t(i)$ and $K_t(i)$ are firm-level employment and capital, respectively.

- Each firm accumulates capital according to the law of motion,

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i) I_t(i), \tag{67}$$

Firms' Labor Choice

- A firm's dividend in period t is hence given by

$$D_t(i) = Y_t(i) - P_t I_t(i) - W_t N_t(i), \tag{68}$$

where $P_t = 1$ denotes the relative price of investment goods and W_t the competitive real wage.

- Denote $n_t(i) \equiv N_t(i)/K_t(i)$ as the labor-capital ratio and $f(\cdot) \equiv F(1, \cdot)$ as the output-capital ratio. Given the real wage, the firm's optimal labor demand is determined by the equation,

$$f_n(A_t n_t(i)) A_t = W_t. \tag{69}$$

Note that the labor demand function implies that all firms choose the same labor-capital ratio, namely, $n_t(i) = n(w_t, A_t)$ for all i.

- Firm i's operating profits can then be expressed as

$$\max_{N_t(i)} \{Y_t(i) - W_t N_t(i)\} = R_t K_t(i), \tag{70}$$

where $R_t \equiv f(A_t n_t) - w_t n_t$

Firms' Labor Choice

We make the following additional assumptions:

- Firms' investment is financed by credit and is subject to the borrowing constraints:
$$I_t(i) \leq \theta K_t(i), \tag{71}$$
where $\theta > 0$ is a constant. This borrowing constraint is similar to the collateralized borrowing assumed by Kiyotaki and Moore (1997).
- Firm-level investment may be partially irreversible:
$$K_{t+1}(i) \geq \rho\left(1-\delta\right) K_t(i), \tag{72}$$
- or
$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)} K_t(i), \tag{73}$$
where $\tilde{\rho} \equiv (1-\rho)(1-\delta)$.

Firms' investment choice

- With the definition in equation (70), a firm's maximization problem can be rewritten as

$$\max_{\{i_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left(R_t K_t(i) - I_t(i) \right) \tag{74}$$

subject to equations

$$I_t(i) \leq \theta K_t(i); I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)} K_t(i) \tag{75}$$

and

$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)} K_t(i), \tag{76}$$

Firms' first order condition

- Denote $\{\lambda_t(i), \pi_t(i), \mu_t(i)\}$ as the Lagrangian multipliers of above constraints , respectively, the firm's first order conditions for $\{I_t(i), K_{t+1}(i)\}$ are given, respectively, by

$$1 = \varepsilon_t(i)\lambda_t(i) + \pi_t(i) - \mu_t(i), \tag{77}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)\lambda_{t+1}(i) + \theta\mu_{t+1}(i) + \tilde{\rho}\frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} \right\}, \tag{78}$$

plus the complementarity slackness conditions,

$$\pi_t(i)\left[I_t(i) + \rho\frac{K_t(i)}{\varepsilon_t(i)}\right] = 0 \tag{79}$$

$$\mu_t(i)[\theta_t K_t(i) - I_t(i)] = 0. \tag{80}$$

Firms' first order condition

- Conjecture the Lagrangian multipliers $\{\lambda_t(i), \pi_t(i), \mu_t(i)\}$ depend only on aggregate states $\mathbf{S}_t$ and $\varepsilon_t(i)$, which implies that the expected value of the Lagrangian multipliers are independent of i; namely, $E_t\lambda_{t+1}(i) = \bar{\lambda}_{t+1}$, $E_t\mu_{t+1}(i) = \bar{\mu}_{t+1}$, and $E_t\pi_{t+1}(i) = \bar{\pi}_{t+1}$. So equation (78) can be rewritten as

$$\lambda_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)\bar{\lambda}_{t+1} + \theta\bar{\mu}_{t+1} + \tilde{\rho} \int \frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} d\Phi(\varepsilon) \right\}, \tag{81}$$

which shows that $\lambda_t(i) = \lambda(\mathbf{S}_t)$ is also independent of i.

Investment Decision Rules

We use a guess-and-verify strategy to derive closed-form decision rules. The decision rules are characterized by a cutoff strategy where the cutoff is defined by the opportunity cost of installing one unit of capital:

$$\lambda_t = \frac{1}{\varepsilon_t^*}. \tag{82}$$

- investment rule is

$$I_t(i) = \begin{cases} \theta K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ -\tilde{\rho}\frac{K_t(i)}{\varepsilon_t(i)} & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \tag{83}$$

- And the two mutilpliers are

$$\mu_t(i) = \max\{q_t(i) - 1, 0\} \tag{84}$$

$$\pi_t(i) = \max\{1 - q_t(i), 0\} \tag{85}$$

$$\frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + \frac{(1-\delta)}{\varepsilon_{t+1}^*} + O(\varepsilon_{t+1}^*) \right\}, \tag{86}$$

where the implicit function

$$\begin{aligned} O(\varepsilon_{t+1}^*) &\equiv E_t \left[\theta \mu_{t+1}(i) + \tilde{\rho} \frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} \right] \\ &= \theta \int_{\varepsilon_{t+1}(i) \geq \varepsilon_{t+1}^*} \frac{\varepsilon_{t+1}(i) - \varepsilon_{t+1}^*}{\varepsilon_{t+1}^*} d\Phi(\varepsilon) \\ &\quad + \tilde{\rho} \int_{\varepsilon_{t+1}(i) < \varepsilon_{t+1}^*} \left(\frac{1}{\varepsilon_{t+1}(i)} - \frac{1}{\varepsilon_{t+1}^*} \right) d\Phi(\varepsilon) \end{aligned} \tag{87}$$

Properties of Aggregate investment Function

- Integrating the firm-level decision rules by the law of large numbers, the aggregate investment, aggregate capital stock, and the optimal cutoff are determined by the following three equations:

$$I_t = \theta K_t [1 - \Phi(\varepsilon_t^*)] - \tilde{\rho} K_t \int_{\varepsilon < \varepsilon^*} \frac{1}{\varepsilon} d\Phi(\varepsilon) \tag{88}$$

$$K_{t+1} = (1 - \delta) K_t + \theta K_t \int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) - \tilde{\rho} K_t \int_{\varepsilon < \varepsilon_t^*} d\Phi(\varepsilon), \tag{89}$$

plus

$$\frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + \frac{(1 - \delta)}{\varepsilon_{t+1}^*} + O(\varepsilon_{t+1}^*) \right\}, \tag{90}$$

Properties of Aggregate investment Function

- By equation (88), the aggregate investment rate is given by

$$\frac{I_t}{K_t} = \theta\left[1 - \Phi(\varepsilon_t^*)\right] - \tilde{\rho}\int_{\varepsilon<\varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon); \tag{91}$$

the above equation also define the cutoff as an implicit function of the investment rate:

$$\varepsilon_t^* = \varepsilon^*(\frac{I_t}{K_t}). \tag{92}$$

- Therefore, equation (89) can be written as

$$K_{t+1} = (1-\delta)K_t + \varphi(\frac{I_t}{K_t})K_t, \tag{93}$$

where $\varphi(\frac{I_t}{K_t}) \equiv \theta\int_{\varepsilon\geq\varepsilon^*(\frac{I_t}{K_t})} \varepsilon d\Phi(\varepsilon) - \tilde{\rho}\int_{\varepsilon<\varepsilon^*(\frac{I_t}{K_t})} d\Phi(\varepsilon)$ is an implicit function of aggregate investment rate.

Properties of Aggregate investment Function

Lemma

The implicit function $\varphi(\cdot)$ is increasing and strictly concave.

Proof.

Denoting $i_t \equiv \frac{I_t}{K_t}$ *and taking derivative of the function* $\varphi(\cdot)$ *with respect to* i_t *gives*

$$\varphi'(i_t) = [-\theta\varepsilon_t^*\phi(\varepsilon_t^*) - \tilde{\rho}\phi(\varepsilon_t^*)]\frac{\partial\varepsilon_t^*}{\partial i_t}, \tag{94}$$

where $\phi(\varepsilon)$ *denotes the PDF of* ε*. Differentiating equation (**??**) with respect to* i_t*, we have*

$$\frac{\partial i_t}{\partial\varepsilon_t^*} = -\theta\phi(\varepsilon_t^*) - \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*). \tag{95}$$

□

Lemma

Proof.

The above two equations together imply

$$\varphi'(i_t) = \frac{\theta\varepsilon_t^*\phi(\varepsilon_t^*) + \tilde{\rho}\phi(\varepsilon_t^*)}{\theta\phi(\varepsilon_t^*) + \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*)} = \varepsilon_t^* > 0. \tag{96}$$

Differentiating this equation with respect to i_t again and using equation (95) gives

$$\varphi''(i_t) = \frac{\partial\varepsilon_t^*}{\partial i_t} = \frac{1}{-\theta\phi(\varepsilon_t^*) - \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*)} < 0. \tag{97}$$

Therefore, the function $\varphi(i_t)$ is increasing and strictly concave in i_t. □

Properties of Aggregate investment Function

- If we redefine the marginal value of newly installed capital as

$$Q_t \equiv \frac{1}{\varepsilon_t^*}, \tag{98}$$

- we can rearrange the implicit function $O(\varepsilon_t^*)$

$$\begin{aligned} O(\varepsilon_t^*) &= \theta \int_{\varepsilon \geq \varepsilon_t^*} \frac{\varepsilon}{\varepsilon_t^*} d\Phi(\varepsilon) - \theta\left[1 - \Phi(\varepsilon_t^*)\right] \\ &\quad + \tilde{\rho} \int_{\varepsilon < \varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon) - \tilde{\rho} \int_{\varepsilon < \varepsilon_t^*} \frac{1}{\varepsilon_t^*} d\Phi(\varepsilon) \\ &= \frac{1}{\varepsilon_t^*}\left[\theta \int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) - \tilde{\rho} \int_{\varepsilon < \varepsilon_t^*} d\Phi(\varepsilon)\right] \\ &\quad - \left\{\theta[1 - \Phi(\varepsilon_t^*)] - \tilde{\rho} \int_{\varepsilon < \varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon)\right\} \\ &= Q_t \varphi(i_t) - i_t. \end{aligned} \tag{99}$$

Properties of Aggregate investment Function

- Therefore, the system of equations that solve for aggregate investment rate (i_t), the capital stock (K_{t+1}), and Q_t are given by

$$Q_t \varphi'(i_t) = 1, \tag{100}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + Q_{t+1}(1-\delta) + Q_{t+1}\varphi(i_{t+1}) - i_{t+1} \right\}, \tag{101}$$

$$K_{t+1} = (1-\delta)K_t + \varphi(i_t)K_t. \tag{102}$$

Equivalence

- Now consider a standard representative-agent macro model of CAC (e.g., Hayashi, 1982), where a representative-firm solves (taking as given the marginal product of capital R_t)

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left(R_t K_t - I_t \right) \tag{103}$$

subject to

$$K_{t+1} = (1-\delta) K_t + \psi(\frac{I_t}{K_t}) K_t. \tag{104}$$

Define Q_t as the Lagrangian multiplier for the constraint (**??**) and $i_t \equiv \frac{I_t}{K_t}$ as the investment rate, the first-order conditions for $\{I_t, K_{t+1}\}$ are given, respectively, by

$$Q_t \psi'(i_t) = 1 \tag{105}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{ R_{t+1} + (1-\delta) Q_{t+1} + Q_{t+1} \left[\psi(i_{t+1}) - \psi'(i_{t+1}) i_{t+1} \right] \}. \tag{106}$$

Using equation (**??**), we can rewrite equation (**??**) as

Equivalence

- Define Q_t as the Lagrangian multiplier for the capital accumulation and $i_t \equiv \frac{I_t}{K_t}$ as the investment rate, the first-order conditions for $\{I_t, K_{t+1}\}$ are given, respectively, by

$$Q_t \psi'(i_t) = 1 \tag{108}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{R_{t+1} + (1-\delta) Q_{t+1} + Q_{t+1} \left[\psi(i_{t+1}) - \psi'(i_{t+1}) i_{t+1}\right]\}. \tag{109}$$

- We can further write

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{R_{t+1} + (1-\delta) Q_{t+1} + Q_{t+1} \psi(i_{t+1}) - i_{t+1}\}. \tag{110}$$

A example

To further illustrate the equivalence result, suppose the distribution of ε is Pareto with support $[1, \infty)$ and shape parameter $\eta > 1$, namely, $\Phi(\varepsilon) = 1 - \varepsilon^{-\eta}$; and assume that investment is completely irreversible, $\tilde{\rho} = 0$. With these assumptions, We have

$$\frac{I_t}{K_t} = \theta \varepsilon_t^{*-\eta} = \theta Q_t^{\eta} \tag{111}$$

$$K_{t+1} = (1-\delta)K_t + \frac{\eta\theta}{\eta - 1}\varepsilon_t^{*1-\eta} K_t. \tag{112}$$

The adjustment cost function function is

$$\varphi(\frac{I_t}{K_t}) = \frac{\eta \theta^{\frac{1}{\eta}}}{\eta - 1}\left(\frac{I_t}{K_t}\right)^{\frac{\eta-1}{\eta}}, \tag{113}$$

The adjustment cost function is

$$\varphi(\frac{I_t}{K_t}) = \frac{\eta\theta^{\frac{1}{\eta}}}{\eta - 1}\left(\frac{I_t}{K_t}\right)^{\frac{\eta-1}{\eta}}, \tag{114}$$

which is homogeneous of degree zero and satisfies

$$\varphi'(\frac{I_t}{K_t})Q_t = 1. \tag{115}$$

And Q_t can be written as

$$Q_t = E_t\beta_{t+1}\left\{R_{t+1} + Q_{t+1}(1-\delta) + \frac{1}{\eta - 1}\frac{I_{t+1}}{K_{t+1}}\right\}; \tag{116}$$

and the law of capital accumulation becomes

$$K_{t+1} = (1-\delta)K_t + \frac{\eta\theta^{\frac{1}{\eta}}}{\eta - 1}I_t^{\frac{\eta-1}{\eta}}K_t^{\frac{1}{\eta}}, \tag{117}$$