Notes on Asset Prices in the production economy-Qtheory

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Household

- A representative household with utility function

\[ E_0 \sum_{t=0}^{\infty} \beta^t [\log(c_t) - A_n \frac{n_t^{1+\gamma}}{1+\gamma}] \]  

(1)

- The constraint:

\[ c_t + V_t s_{t+1} + \frac{b_{t+1}}{R_{ft}} = (V_t + D_t) s_t + w_t n_t + b_t \]  

(2)

- Notation: where \( c_t \) is consumption, \( w_t \) the really wage, \( n_t \) working hours, \( V_t \) the value of a standard share of the firm after dividend payment, and \( D_t \) is the dividend per share. The first order condition with respective to \( n_t \) and \( s_{t+1}, b_{t+1} \) are
Household

- The foc

\[
\frac{1}{c_t} = \lambda_t \tag{3}
\]

\[
\lambda_t w_t = A_n n_t^\gamma \tag{4}
\]

\[
\frac{1}{R_{ft}} \lambda_t = \beta E_t \lambda_{t+1} \tag{5}
\]

\[
V_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [V_{t+1} + D_{t+1}] \tag{6}
\]

- Transversality conditions:

\[
\lim_{j \to \infty} E_t \beta^j \lambda_{t+j} V_{t+j} s_{t+j+1} = 0 \tag{7}
\]

\[
\lim_{j \to \infty} E_t \beta^j \lambda_{t+j} \frac{b_{t+j+1}}{R_{ft+j}} = 0 \tag{8}
\]
Iterating forward we obtain

\[ V_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} + \lim_{j \to \infty} E_t \beta^j \frac{\lambda_{t+j}}{\lambda_t} V_{t+j} \]  \hspace{1cm} (9)

In equilibrium \( s_{t+1} = 1, b_{t+1} = 0 \), so we have

\[ V_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \]  \hspace{1cm} (10)

the value of each share before the dividend payment is

\[ V_t + D_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \]  \hspace{1cm} (11)
The firms

- The representative firm’s problem is maximize the expected discounted dividend or \((V_t + D_t)\)

\[
V_t + D_t = \max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j}
\]  

(12)

- The firm hires labor \(n_t\) from the household, accumulate capital, decides dividend payment with the following flow of fund constraint

\[
D_t + k_{t+1} - (1 - \delta) k_t = A_t k_t^\alpha n_t^{1-\alpha} - w_t n_t
\]  

(13)

- Set the bellman equation

\[
W_t(k_t, A_t) = \max D_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} W_{t+1}(k_{t+1}, A_{t+1})
\]  

(14)
The firms' foc

- The first order condition with respective to \( n_t \), are

\[
\omega_t n_t = (1 - \alpha) y_t \quad (15)
\]

- and

\[
1 = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \frac{\partial W_{t+1}(k_{t+1}, A_{t+1})}{\partial k_{t+1}} \quad (16)
\]

- Envelop theory:

\[
\frac{\partial W_t(k_t, A_t)}{\partial k_t} = \alpha y_t / k_t + (1 - \delta) \quad (17)
\]

- so we have

\[
\lambda_t = \beta E_t \lambda_{t+1} [\alpha y_{t+1} / k_{t+1} + (1 - \delta)] \quad (18)
\]
The firms’ value after dividend payment

- We have obtained

\[ \lambda_t = \beta E_t \lambda_{t+1} [\alpha y_{t+1} / k_{t+1} + (1 - \delta)] \]  

(19)

- Multiply \( k_{t+1} \) to both side

\[ \lambda_t k_{t+1} = \beta E_t \lambda_{t+1} [\alpha y_{t+1} + (1 - \delta) k_{t+1}] \]  

(20)

- Use the law of capital

\[ k_{t+2} = (1 - \delta) k_{t+1} + l_t \]  

(21)

- we have

\[ \lambda_t k_{t+1} = \beta E_t \lambda_{t+1} [\alpha y_{t+1} - l_{t+1} + k_{t+2}] \]  

(22)

\[ = \beta E_t \lambda_{t+1} [D_{t+1} + k_{t+2}] \]
The firms’ value after dividend payment

- This implies

\[ k_{t+1} = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} = V_t \]  

(23)

- Hence the value of firm before devident is just \( k_{t+1} \)

\[ k_{t+1} + D_t = \alpha y_t + (1 - \delta) k_t \]  

(24)

- Since

\[ w_t = (1 - \alpha) A_t k_t^\alpha n_t^{-\alpha} \rightarrow n_t = \left( \frac{(1 - \alpha) A_t}{w_t} \right) \frac{1}{\alpha} k_t \]  

(25)

- so we have

\[ y_t = A_t \left( \frac{(1 - \alpha) A_t}{w_t} \right)^{\frac{1-\alpha}{\alpha}} k_t \]  

(26)

Then we have \( \alpha y_t + (1 - \delta) k_t \) is proportional to firm’s capital stock.
Equilibrium Condition

Equilibrium is defined in the usual way. Namely given the prices, household maximizes his utility and firm maximizes its objective function.

- The budget constraint:
  \[ c_t = D_t + w_t n_t = y_t - (k_{t+1} - (1 - \delta)k_t) - w_t n_t + w_t n_t \]  
  \[ (27) \]

- or we have
  \[ c_t + k_{t+1} = y_t + (1 - \delta)k_t \]  
  \[ (28) \]

- The asset value
  \[ \frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} [\alpha y_{t+1} / k_{t+1} + (1 - \delta)] \]  
  \[ (29) \]

- and the first order condition with labor
  \[ \frac{1}{c_t} w_t = A_n n_t^\gamma \]  
  \[ (30) \]

- The production function
  \[ y_t = A_t k_t^\alpha n_t^{1-\alpha} \]  
  \[ (31) \]
Verify the model have the same equilibrium as the following planner’s problem:

- **Utility function**

\[ E_0 \sum_{t=0}^{\infty} \beta^t [\log(c_t) - A_n \frac{n_t^{1+\gamma}}{1 + \gamma}] \]  

(32)

- **Constraint:**

\[ c_t + k_{t+1} = y_t + (1 - \delta) k_t \]  

(33)

- **Production**

\[ y_t = A_t k_t^{\alpha} n_t^{1-\alpha} \]  

(34)
Investment adjustment cost and the Q-theory

The household’s problem is the same as before. The firm’s problem is changed. We assume the firm need to pay investment adjustment cost.

More specifically a firm needs to pay $I_t[1 + \varphi(\frac{I_t}{K_t})]$ to increase capital by $I_t$ units. The firm’s objective function is still to maximize the max

$$
\max_t E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \iff \max_t E_t \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} D_{t+j}
$$

(35)

The cash-flow of the firm changes to

$$
D_t + I_t[1 + \varphi(\frac{I_t}{K_t})] = A_t k_t^\alpha n_t^{1-\alpha} - w_t n_t
$$

(\lambda)

And capital law of motion

$$
k_{t+1} = (1 - \delta)k_t + I_t
$$

(\mu)
The first order condition with respect to $I_t, k_{t+1}$ yields

$$\lambda_t [1 + \phi\left(\frac{I_t}{k_t}\right) + \frac{I_t}{k_t} \phi'(\frac{I_t}{k_t})] = \mu_t$$  \hspace{1cm} (36)$$

and

$$\mu_t = \beta E_t \{ (1 - \delta) \mu_{t+1} + \lambda_{t+1} \left[ \frac{\alpha y_{t+1}}{k_{t+1}} + \frac{I_{t+1}^2}{k_{t+1}^2} \phi'(\frac{I_{t+1}}{k_{t+1}}) \right] \}$$ \hspace{1cm} (37)$$

Define a variable $q_t = \frac{\mu_t}{\lambda_t}$, the above two equations can be expressed as

$$1 + \phi\left(\frac{I_t}{k_t}\right) + \frac{I_t}{k_t} \phi'(\frac{I_t}{k_t}) = \frac{\mu_t}{\lambda_t} = q_t$$ \hspace{1cm} (38)$$

This is the marginal value of installed capital this period, which evolves according

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [(1 - \delta) q_{t+1} + \frac{\alpha y_{t+1}}{k_{t+1}} + \frac{I_{t+1}^2}{k_{t+1}^2} \phi'(\frac{I_{t+1}}{k_{t+1}})]$$ \hspace{1cm} (39)$$
To get a better understanding of $q_t$, take a special form of 
\[ \varphi\left(\frac{l_t}{k_t}\right) = b \frac{l_t}{k_t}, \]
we then have
\[ 2b \frac{l_t}{k_t} = q_t - 1 \quad (40) \]
or
\[ l_t = \frac{(q_t - 1)}{2b} k_t \quad (41) \]
we then have $l_t > 0$ if and only if $q_t > 1$.

To increase one unit of installed capital, the firm needs to sacrifice 
\[ 1 + 2b \frac{l_t}{k_t}, \]
so this is the marginal cost of installed capital. What is the gain? The cost-benefit analysis implies the marginal gain must equal to the marginal cost. So the gain would be $q_t$. 
The value of firm in the example

- Using the equation for $q_t$, and the expression for $I_t$, we have

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \left[(1 - \delta)q_{t+1} + \frac{\alpha y_{t+1}}{k_{t+1}} + \frac{I_{t+1}^2}{k_{t+1}^2} \varphi'(\frac{I_{t+1}}{k_{t+1}})\right] \quad (42)$$

- Multiply both sides by $k_{t+1}$ and recall $\varphi'(\frac{I_{t+1}}{k_{t+1}}) = b$

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [(1 - \delta)q_{t+1}k_{t+1} + \alpha y_{t+1} + \frac{I_{t+1}^2}{k_{t+1}} b] \quad (43)$$

- to get obtain the firm value, recall

$$(1 - \delta)k_{t+1} = k_{t+2} - I_{t+1} \quad (44)$$
The value of firm in the example

- to get obtain the firm value, recall

\[(1 - \delta)k_{t+1} = k_{t+2} - l_{t+1}\]  (45)

- and

\[q_t = 2b \frac{l_t}{k_t} + 1\]  (46)

- so we have Hence

\[(1 - \delta)q_{t+1}k_{t+1} = q_{t+1}k_{t+2} - q_{t+1}l_{t+1}\]

\[= q_{t+1}k_{t+2} - (2b \frac{l_{t+1}}{k_{t+1}} + 1)l_{t+1}\]  (47)

- And

\[q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [(1 - \delta)q_{t+1}k_{t+1} + \alpha y_{t+1} + \frac{l_{t+1}^2}{k_{t+1}^2} b]\]
The value of firm in the example

- We now have

\[
q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [(1 - \delta) q_{t+1} k_{t+1} + \alpha y_{t+1} + \frac{l_{t+1}^2}{k_{t+1}} b] \\
= \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [q_{t+1} k_{t+2} + \alpha y_{t+1} - l_{t+1}(1 + b \frac{l_{t+1}}{k_{t+1}})] \\
= \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [q_{t+1} k_{t+2} + D_{t+1}] 
\]

(48)

- so we have

\[
q_t k_{t+1} + D_t = V_t + D_t 
\]

(49)
Labor Decision

- Consider that the production function is

\[ y_t = A_t F(k_t, n_t) \]  \hspace{1cm} (50)

- where \( F(k_t, n_t) \) exhibits constant return to scale. Given \( k_t \), the decision for \( n_t \) is static.

- By CRS we can write

\[ \max A_t F(k_t, n_t) - w_t n_t = \max_{\tilde{n}_t} [A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t] k_t \]  \hspace{1cm} (51)

where \( \tilde{n}_t = \frac{n_t}{k_t} \). The above equation yields

\[ A_t F_2(1, \tilde{n}_t) = w_t \]  \hspace{1cm} (52)

- This implies that \( A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t \) would be a function of \( w_t \) and \( A_t \), which does not depend on capital stock. We denote

\[ A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t = R_t \]
Notice by

\[ A_t F_2(1, \tilde{n}_t) = w_t \]  \hspace{1cm} (53)

the labor-capital ratio does not depend on firm’s capital, \( k_t \), it only depends on wage and technology level.

Hence

\[ R_t = A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t \]  \hspace{1cm} (54)

is independent of firm’s capital stock too.
We now prove

\[ R_t = AF'(K_t, N_t) \]  \hspace{1cm} (55)

where \( K_t \) and \( N_t \) is the aggregate capital and labor. Notice in equilibrium \( k_t = K_t, n_t = N_t \).

first by CRS

\[ A_t F(1, \frac{n_t}{k_t}) k_t = A_t F(k_t, n_t) \]  \hspace{1cm} (56)

so we have

\[ w_t = A_t F_2(1, \frac{n_t}{k_t}) = A_t F_2(k_t, n_t) = A_t F_2(K_t, N_t) \]  \hspace{1cm} (57)

Take derivative with respect to \( k_t \) we have

\[ A_t F(1, \frac{n_t}{k_t}) - A_t F_2(1, \frac{n_t}{k_t}) \frac{n_t}{k_t} = A_t F_1(k_t, n_t) = A_t F_1(K_t, N_t) \]  \hspace{1cm} (58)

or

\[ R_t = A_t F(1, \tilde{n}_t) - A_t F_2(1, \tilde{n}_t) \tilde{n}_t = A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t = A_t F_1(K_t, N_t) \]  \hspace{1cm} (59)
Cash flow

- with \( R_t \), we have

\[
d_t + I_t [1 + \varphi \left( \frac{I_t}{k_t} \right)] = R_t k_t
\] (60)

The firm objective is

\[
P_t = \max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} = D_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} P_{t+1}
\]

\[
= \max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} \left[ R_t k_t - I_t [1 + \varphi \left( \frac{I_t}{k_t} \right)] \right]
\]

subject to

\[
k_{t+1} = (1 - \delta) k_t + I_t
\] (61)
define the investment rate as

\[ I_t = i_t \times k_t \]  \hfill (62)

And guessing

\[
E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} \left[ R_t k_t - I_t \left[ 1 + \varphi \left( \frac{I_t}{k_t} \right) \right] \right] = p_t k_t
\]

by definition we then must have

\[
p_t k_t = \max \left[ R_t k_t - I_t \left[ 1 + \varphi \left( \frac{I_t}{k_t} \right) \right] + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} k_{t+1} \right] \]

\[ = \max \left[ R_t k_t - I_t \left[ 1 + \varphi \left( \frac{I_t}{k_t} \right) \right] + \beta (1 - \delta + i_t) k_t E_t \frac{\lambda_{t+1}}{\lambda_t} p_t \right] \]

\hfill (65)
or we have

\[ p_t = \max\left[R_t - i_t[1 + \varphi(i_t)] + \beta(1 - \delta + i_t)E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1}\right] \] \hspace{1cm} (66)

Define \( E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} = q_t \), so we have

\[ \varphi'(i_t)i_t + \varphi(i_t) = [q_t - 1] \] \hspace{1cm} (67)

And value of firm after dividend is

\[ \beta(1 - \delta + i_t)k_t E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} = q_t k_{t+1} \] \hspace{1cm} (68)

By definition we have

\[ q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} \] \hspace{1cm} (69)
the firm’s value

Now we need to calculate $p_t$, by definition we have

$$p_t = R_t - i_t[1 + \phi(i_t)] + i_t q_t + (1 - \delta) q_t \quad (70)$$

since $R_t = A_t F_1(k_t, n_t)$ and $q_t = \phi'(i_t)i_t + \phi(i_t) + 1$, we have

$$p_t = R_t + (1 - \delta) q_t - i_t[1 + \phi(i_t)] + i_t q_t$$
$$= R_t + (1 - \delta) q_t - i_t[1 + \phi(i_t)] + [1 + \phi'(i_t)i_t + \phi(i_t)] i_t$$
$$= R_t + (1 - \delta) q_t + \phi'(i_t)i_t^2 \quad (71)$$

Hence we have

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [A_{t+1} F_1(k_{t+1}, n_{t+1}) + (1 - \delta) q_{t+1} + \phi'(\frac{l_{t+1}}{k_{t+1}}) \frac{l_{t+1}^2}{k_{t+1}^2}] \quad (72)$$
the firm’s value

- multiply both sides by $k_{t+1}$ we have

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [(1 - \delta) q_{t+1} k_{t+1} + A_{t+1} F_1(k_{t+1}, n_{t+1}) k_{t+1}$$

$$+ \frac{l_{t+1}^2}{k_{t+1}} \varphi'(\frac{l_{t+1}}{k_{t+1}})]$$

(73)

- using

$$q_{t+1} = 1 + \varphi'(\frac{l_{t+1}}{k_{t+1}}) \frac{l_{t+1}}{k_{t+1}} + \varphi(\frac{l_{t+1}}{k_{t+1}})$$

(74)

and

$$(1 - \delta) k_{t+1} = k_{t+2} - l_{t+1}$$

(75)
the firm’s value

- We can rewrite

\[(1 - \delta)q_{t+1}k_{t+1} + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1} + \frac{l_{t+1}^2}{k_{t+1}} \varphi'(\frac{l_{t+1}}{k_{t+1}}) (76)\]

\[= q_{t+1}k_{t+2} - q_{t+1}l_t + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1} + \frac{l_{t+1}^2}{k_{t+1}} \varphi'(\frac{l_{t+1}}{k_{t+1}}) (77)\]

- recall

\[q_{t+1} = 1 + \varphi'(\frac{l_{t+1}}{k_{t+1}}) \frac{l_{t+1}}{k_{t+1}} + \varphi\left(\frac{l_{t+1}}{k_{t+1}}\right) (77)\]

- so

\[q_{t+1}k_{t+2} - q_{t+1}l_t + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1} + \frac{l_{t+1}^2}{k_{t+1}} \varphi'(\frac{l_{t+1}}{k_{t+1}}) (78)\]

\[= q_{t+1}k_{t+2} - l_{t+1} - \varphi\left(\frac{l_{t+1}}{k_{t+1}}\right) l_{t+1} + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1}\]
Recall

\[ A_{t+1} F_1(k_{t+1}, n_{t+1}) k_{t+1} = A_t F(k_{t+1}, n_{t+1}) - w_{t+1} n_{t+1} \] \hspace{1cm} (79)

\[ D_{t+1} = A_t F(k_{t+1}, n_{t+1}) - w_{t+1} n_{t+1} - l_{t+1} - \varphi(\frac{l_{t+1}}{k_{t+1}}) l_{t+1} \] \hspace{1cm} (80)

so we have

\[ q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [q_{t+1} k_{t+2} + D_{t+1}] \] \hspace{1cm} (81)

we established the proof.
different form of capital adjustment cost

- The firm’s objective function is

\[
\max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_t
\]  

\[ (82) \]

- the dividend is

\[ D_t = A_t F(k_t, n_t) - w_t n_t - I_t \]  

\[ (83) \]

- substitute the optimal labor choice as before

\[ \max_{n_t} A_t F(k_t, n_t) - w_t n_t = R_t k_t \]  

\[ (84) \]
different form of capital adjustment cost

And guessing the value function for the firm is linear

$$\max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_t = p_t k_t$$  \hspace{1cm} (85)

by definition we have

$$p_t k_t = \max[R_t k_t - l_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} k_{t+1}]$$  \hspace{1cm} (86)

$$= \max[R_t k_t - l_t + \beta (1 - \delta + \varphi(\frac{l_t}{k_t})) k_t E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1}]$$  \hspace{1cm} (87)
different form of capital adjustment cost

- as before define $\frac{l_t}{k_t} = i_t$, we have

$$p_t k_t = \max [R_t k_t - l_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} k_{t+1}]$$  \hspace{1cm} (88)

$$= k_t \max [R_t - i_t + \beta (1 - \delta + \varphi(i_t)) E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1}]$$  \hspace{1cm} (89)

- define $\beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} = q_t$. We have

$$1 = q_t \varphi'(i_t)$$  \hspace{1cm} (90)

- hence we have

$$p_t = R_t - i_t + q_t (1 - \delta + \varphi(i_t))$$  \hspace{1cm} (91)
To obtain $q_t$, we have

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [R_{t+1} - i_{t+1} + q_{t+1}(1 - \delta + \varphi(i_{t+1}))]$$  \hspace{1cm} (92)

or

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [R_{t+1} k_{t+1} - l_{t+1} + q_{t+1}(1 - \delta + \varphi(\frac{l_{t+1}}{k_{t+1}})) k_{t+1}]$$  \hspace{1cm} (93)

or we have

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [D_{t+1} + q_{t+1} k_{t+2}]$$  \hspace{1cm} (94)

so we have

$$V_t = q_t k_{t+1}$$  \hspace{1cm} (95)
Now consider the centralized version with a special utility:

$$u_t = \log(c_t - h c_{t-1}) - a_L n_t$$

- There are $c_t, k_t, A_t$, three state variables
- There are two constraints

$$c_t + i_t = y_t$$

$$k_{t+1} = (1 - \delta) k_t + \varphi \left( \frac{i_t}{k_t} \right) k_t$$ (96)

- Denote $\lambda_t, \chi_t$ are the Lagrangian multipliers of the above two constraints. Denote $q_t = \frac{\chi_t}{\lambda_t}$, we assume

$$y_t = A_t k_t^\alpha n_t^{1-\alpha}$$
So we set up the Bellman Equations as

\[
V(c_{t-1}, k_t, A_t) = \max_{c_t, n_t, i_t, k_{t+1}} \left\{ \frac{(c_t - hc_{t-1})^{1-\gamma}}{1 - \gamma} - a_L n_t \right\} + \beta E_t V(c_t, k_{t+1}, A_{t+1}) + \lambda_t [A_t k_t^{\alpha} n_t^{1-\alpha} - c_t - i_t] + \lambda_t q_t [(1 - \delta) k_t + \varphi(\frac{i_t}{k_t}) k_t - k_{t+1}] 
\]
Model setup-FOCs

- first order condition with respect to $c_t$:

$$
(c_t - h c_{t-1})^{-\gamma} + \beta E_t V'_c(c_t, k_{t+1}, A_{t+1}) = \lambda_t
$$

(99)

- first order condition with respect to $n_t$:

$$
\lambda_t \frac{(1 - \alpha) y_t}{n_t} = a_L
$$

(100)

- first order condition with respect to $i_t$:

$$
\lambda_t = \lambda_t q_t \varphi'\left(\frac{i_t}{k_t}\right)
$$

(101)
Model setup-FOCs

- first order condition with respect to \( k_{t+1} \):

\[
\lambda_t q_t = \beta E_t V'_k(c_t, k_{t+1}, A_{t+1})
\]  
(102)

- Envelop Theory

\[
V'_c(c_{t-1}, k_t, A_t) = -h(c_t - hc_{t-1})^{-\gamma}
\]  
(103)

\[
V'_k(c_{t-1}, k_t, A_t) = \lambda_t \frac{\alpha y_t}{k_t} + \lambda_t q_t [(1 - \delta) + \varphi\left(\frac{i_t}{k_t}\right) - \varphi'\left(\frac{i_t}{k_t}\right)\frac{i_t}{k_t}]
\]  
(104)
So the first order conditions are be summarized by

$$\lambda_t = (c_t - hct_{-1})^{-\gamma} - \beta hE_t (c_{t+1} - hct)^{-\gamma} \quad (105)$$

$$\lambda_t \frac{(1 - \alpha)y_t}{n_t} = a_L \quad (106)$$

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \left\{ \frac{\alpha y_{t+1}}{k_{t+1}} + q_{t+1} \left[ (1 - \delta) + \varphi\left( \frac{i_{t+1}}{k_{t+1}} \right) - \varphi'(\frac{i_{t+1}}{k_{t+1}}) \frac{i_{t+1}}{k_{t+1}} \right] \right\} \quad (107)$$

$$1 = q_t \varphi'(\frac{i_t}{k_t}) \quad (108)$$

$$y_t = A_t k_t^\alpha n_t^{1-\alpha} = c_t + i_t \quad (109)$$

and capital follows

$$k_{t+1} = (1 - \delta)k_t + \varphi\left( \frac{i_t}{k_t} \right) k_t \quad (110)$$
Equivalence

- The only difference between the decentralized version and plan’er problem is equation regarding $q_t$

\[
q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \left\{ \alpha y_{t+1} \frac{\alpha y_{t+1}}{k_{t+1}} + q_{t+1}[(1 - \delta) + \varphi\left(\frac{i_{t+1}}{k_{t+1}}\right) - \varphi'\left(\frac{i_{t+1}}{k_{t+1}}\right)\frac{i_{t+1}}{k_{t+1}}] \right\}
\]

(111)

\[
q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \left[ R_{t+1} - i_{t+1} + q_{t+1}(1 - \delta + \varphi(i_{t+1})) \right]
\]

(112)

- We now prove these two are the same. Notice

\[
1 = q_t \varphi'\left(\frac{i_t}{k_t}\right)
\]

(113)

- so we have

\[
-i_{t+1} = -i_{t+1} q_{t+1} \varphi'\left(\frac{i_{t+1}}{k_{t+1}}\right)
\]

(114)
Recall the risk free rate is

$$\lambda_t = \beta R_{ft} E_t \lambda_{t+1}$$  \hspace{1cm} (115)

and the stock price

$$\lambda_t V_t = \beta E_t \lambda_{t+1} [V_{t+1} + D_{t+1}]$$  \hspace{1cm} (116)

or

$$\lambda_t = \beta E_t \lambda_{t+1} \left[ \frac{V_{t+1} + D_{t+1}}{V_t} \right] = \beta E_t \lambda_{t+1} R_{et+1}$$  \hspace{1cm} (117)
The log linearize return

\[ 1 = \beta E_t \exp(\log \lambda_{t+1} - \log \lambda_t + r_{ft}) \]  \hspace{1cm} (118)

and the stock price

\[ 1 = \beta E_t [\log \lambda_{t+1} - \log \lambda_t + \log(V_{t+1} + D_{t+1}) - \log V_t] \]  \hspace{1cm} (119)

or

\[ 1 = \beta E_t \exp(\log \lambda_{t+1} - \log \lambda_t + r_{et+1}) \]  \hspace{1cm} (120)

we are interested in the risk premium

\[ \Delta r = E_t r_{et+1} - r_{ft} \]  \hspace{1cm} (121)
The log-linearization around the deterministic steady-state. In the deterministic steady-state, there is no uncertainty we must have

\[ r_f = r_e = -\log(\beta) \]  

(122)

so the above two equations implies

\[ E_t \hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_{ft} = 0 \]  

(123)

and

\[ E_t \hat{\lambda}_{t+1} - \hat{\lambda}_t + E_t \hat{r}_{et+1} = 0 \]  

(124)

The premium is

\[ E_t r_{et+1} - r_{ft} = E_t r_{et+1} - r_e + r_e - r_{ft} \]

\[ = E_t \hat{r}_{et+1} - \hat{r}_{ft} \]

\[ = 0 \]
We have see that directly using log-linearization generate zero risk premium. So we need to look at some other methods.

We now look at the second order approximation. For $R_{ft}$, define $\bar{R}_f = ER_{ft}$ as the unconditional mean of risk free rate and $\bar{R}_e = ER_{et}$ similary. We have

$$R_{ft} = \frac{1}{\beta E_t \exp(\log \lambda_{t+1} - \log \lambda_t)}$$

$$= \frac{1}{\beta E_t \exp(\log \lambda_{t+1} - \log \lambda - (\log \lambda_t - \log \lambda))}$$

$$= \frac{1}{\beta E_t \exp(-\hat{\lambda}_{t+1} + \hat{\lambda}_t)}$$

(125)
Indirect log-linearization

- for real variable we have

\[
\hat{x}_t = \pi_k \hat{k}_t + \pi_A \hat{A}_t
\]  

(126)

\[
\hat{k}_{t+1} = \pi_k \hat{k}_t + \pi_A \hat{A}_t
\]

- where

\[
\hat{A}_t = \rho \hat{A}_t + \varepsilon_t
\]  

(127)

- where \(\varepsilon_t\) is a normal distribution with standard deviation \(\sigma\). Iteration we can write

\[
\hat{k}_{t+1} = \frac{\pi_k \hat{A}_t}{1 - \pi_k L} = \frac{\pi_k}{1 - \pi_k L} \frac{\varepsilon_t}{1 - \rho L}
\]  

(128)

- so

\[
\hat{x}_t = \frac{\pi_k \pi_A}{1 - \pi_k L} \frac{L \varepsilon_t}{1 - \rho L} + \frac{\pi_A \varepsilon_t}{1 - \rho L}
\]
Indirect log-linearization

We have

$$\hat{x}_t = \frac{\pi_k \pi_A^x}{1 - \pi_k^k L} \frac{L \epsilon_t}{1 - \rho L} + \frac{\pi_A^x \epsilon_t}{1 - \rho L}$$  

(129)

It implies we can write

$$\hat{x}_t = \sum_{j=0}^{\infty} \pi_j^x \epsilon_{t-j}$$  

(130)

so the log difference between a variable $x_t$ and its deterministic steady-state are normally distributed.
Indirect log-linearization

- Using the above fact we have

\[ \hat{\lambda}_t = \sum_{j=0}^{\infty} \pi_{\lambda,j} \varepsilon_{t-j} \]  \hspace{1cm} (131)

- and

\[ \hat{\lambda}_{t+1} = \sum_{j=0}^{\infty} \pi_{\lambda,j} \varepsilon_{t+1-j} \]  \hspace{1cm} (132)

- The expected value

\[ E_t \hat{\lambda}_{t+1} = \sum_{j=0}^{\infty} \pi_{\lambda,j+1} \varepsilon_{t-j} \]  \hspace{1cm} (133)

- Conditional variance

\[ \text{Var}_t(\hat{\lambda}_{t+1}) = \pi_{\lambda,0}^2 \sigma^2 \]  \hspace{1cm} (134)
Indirect log-linearization

- So the implied risk free rate is

\[ R_{ft} = \frac{1}{\beta} \frac{1}{E_t \exp\{\hat{\lambda}_t - \hat{\lambda}_{t+1}\}} \]
\[ = \frac{1}{\beta} \exp\{-E_t \hat{\lambda}_{t+1} + \hat{\lambda}_t - \frac{1}{2} \text{var}_t(\hat{\lambda}_{t+1})\} \]
\[ = \frac{1}{\beta} \exp\{\sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1}) \varepsilon_{t-j} - \frac{1}{2} \pi_{\lambda,0}^2 \sigma^2\} \]  
(135)

- we take log and we then have

\[ r_{ft} = -\log \beta + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1}) \varepsilon_{t-j} - \frac{1}{2} \pi_{\lambda,0}^2 \sigma^2 \]
(137)

- so the unconditional mean is

\[ Er_{ft} = -\log \beta - \frac{1}{2} \pi_{\lambda,0}^2 \sigma^2 \]
(138)

- which is different from the deterministic steady-state.
Indirect log-linearization

- To obtain the unconditional mean of $R_{ft}$ we use

$$R_{ft} = \frac{1}{\beta} \exp\left\{ \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1}) \varepsilon_{t-j} - \frac{1}{2} \pi_{\lambda,0}^2 \sigma^2 \right\}$$  \hspace{1cm} (139)

- Again $\sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1}) \varepsilon_{t-j}$ has normal distribution we have

$$ER_{ft} = \bar{R}_f = \frac{1}{\beta} \exp\left( -\frac{1}{2} \pi_{\lambda,0}^2 \sigma^2 + \frac{1}{2} \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2 \sigma^2 \right)$$  \hspace{1cm} (140)

- The volatility of risk free rate

$$r_{ft} = -\log \beta + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1}) \varepsilon_{t-j} + \frac{1}{2} \pi_{\lambda,0}^2 \sigma^2$$  \hspace{1cm} (141)

- so we have

$$\text{var}(r_{ft}) = \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2 \sigma^2$$  \hspace{1cm} (142)
The risk return

\[ 1 = \beta E_t \exp\{\hat{\lambda}_{t+1} - \hat{\lambda}_t + r_{et+1}\} \]  \hspace{1cm} (143)

Define \( \bar{R}_e = E_t R_{et} \)

\[ 1 = \beta E_t \exp\{\hat{\lambda}_{t+1} - \hat{\lambda}_t + r_{et+1}\} \]
\[ = \beta \bar{R}_e E_t \exp\{\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_{et+1}\} \]  \hspace{1cm} (144)
Indirect log-linearization of risky return

- Using the property of normal distribution, the term \( \hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_{et+1} \) is also normally distributed.

- Its mean

\[
E_t[\hat{\lambda}_{t+1} - \hat{\lambda}_t] = \sum (\pi_{\lambda,j+1} - \pi_{\lambda,j} + \pi_{r,j}) \varepsilon_{t-j} (145)
\]

- Its variance is

\[
Var_t(\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_{et+1}) = Var_t(\pi_{\lambda,0} \varepsilon_{t+1} + \pi_{r,0} \varepsilon_{t+1})
= (\pi_{\lambda,0}^2 + \pi_{r,0}^2 + 2\pi_{\lambda,0} \pi_{r,0}) \sigma^2 (146)
\]

- So we have

\[
\bar{R}_e = \frac{1}{\beta} \exp \left[ \sum \left( -\pi_{\lambda,j+1} + \pi_{\lambda,j} - \pi_{r,j+1} \right) \varepsilon_{t-j} - \frac{1}{2} (\pi_{\lambda,0}^2 + \pi_{r,0}^2 + 2\pi_{\lambda,0} \pi_{r,0}) \sigma^2 \right] (147)
\]
Indirect log-linearization of risky return

- or we have

$$\log \bar{R}_e = - \log \beta - \sum (\pi_{\lambda,j+1} - \pi_{\lambda,j} + \pi_{r,j+1}) \varepsilon_{t-j} - \frac{1}{2} \left( \pi_{\lambda,0}^2 + \pi_{r,0}^2 + 2 \pi_{\lambda,0} \pi_{r,0} \right) \sigma^2$$

(148)

- so we must have

$$\pi_{\lambda,j+1} - \pi_{\lambda,j} + \pi_{r,j+1} = 0$$

(149)

- or we have

$$\log \bar{R}_e = - \log \beta - \frac{1}{2} \left( \pi_{\lambda,0}^2 + \pi_{r,0}^2 + 2 \pi_{\lambda,0} \pi_{r,0} \right) \sigma^2$$

(150)

- and the risk rate is

$$\log \bar{R}_f = - \log \beta + \frac{1}{2} \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2 \sigma^2 - \frac{1}{2} \pi_{\lambda,0}^2 \sigma^2$$

(151)
Indirect log-linearization of risky return

The premium

\[
\log \bar{R}_e = - \log \beta - \frac{1}{2} (\pi^2_{\lambda,0} + \pi^2_{r0} + 2\pi_{\lambda,0} \pi_{r,0}) \sigma^2
\]  \hspace{1cm} (152)

\[
\log \bar{R}_f = - \log \beta - \frac{1}{2} \pi^2_{\lambda,0} \sigma^2 + \frac{1}{2} \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2 \sigma^2
\]  \hspace{1cm} (153)

or we have

\[
\log \bar{R}_e - \log \bar{R}_f = - \pi_{\lambda,0} \pi_{r,0} \sigma^2 - \frac{1}{2} (r^2_{r0} + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2) \sigma^2
\]  \hspace{1cm} (154)
Indirect log-linearization of risky return

- we have

\[
\log \tilde{R}_e - \log \tilde{R}_f = -\pi_{\lambda,0} \pi_{r,0} \sigma^2 - \frac{1}{2} \left( \pi_{r0}^2 + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2 \right) \sigma^2
\]

(155)

- Finally by

\[
\pi_{\lambda,j+1} - \pi_{\lambda,j} + \pi_{r,j+1} = 0
\]

(156)

- we can write

\[
\pi_{r0}^2 + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2 = \pi_{r0}^2 + \sum_{j=0}^{\infty} \pi_{r,j+1}^2 = \sum_{j=0}^{\infty} \pi_{r,j} \sigma^2
\]

(157)

- The unconditional volatility of

\[
\text{var}(\hat{r}_{et}) = \text{var}(r_{et}) = \text{var}\left(\sum_{j=0}^{\infty} \pi_{r,j} \varepsilon_{t-j}\right) = \sum_{j=0}^{\infty} \pi_{r,j}^2 \sigma^2
\]

(158)
A special case

- Consider a special case, with $h = 0$. So we have

$$\hat{\lambda}_t = -\gamma \hat{c}_t$$  \hspace{1cm} (159)

- so we have

$$\hat{\lambda}_t = -\gamma \sum \pi_{c,j} \varepsilon_{t-j}$$  \hspace{1cm} (160)

- the consumption growth is

$$\hat{c}_{t+1} - \hat{c}_t = \pi_{c0} \varepsilon_{t+1} + \sum_{j=0}^{\infty} (\pi_{c,j+1} - \pi_{\lambda,j}) \varepsilon_{t-j}$$  \hspace{1cm} (161)

- The covariance between consumption growth and risky return is

$$cov_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) = cov_t(\pi_{c0} \varepsilon_{t+1}, \pi_{r0} \varepsilon_{t+1}) = \pi_{c0} \pi_{r0} \sigma^2$$  \hspace{1cm} (162)
A special case

• The covariance between consumption growth and risky return is

\[ \text{cov}_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) = \text{cov}_t(\pi c_0 \varepsilon_{t+1}, \pi r_0 \varepsilon_{t+1}) = \pi c_0 \pi r_0 \sigma^2 \] (163)

• So the risk premium can be written as

\[
\log \bar{R}_e - \log \bar{R}_f = -\pi_{\lambda,0} \pi_{r,0} \sigma^2 - \frac{1}{2} (\pi_{r0}^2 + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2) \sigma^2 \\
= \gamma \pi c_0 \pi r_0 \sigma^2 - \frac{1}{2} (\pi_{r0}^2 + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2) \sigma^2 \\
= \gamma \text{cov}_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) - \frac{1}{2} \text{var}_t(r_{et+1})
\]
A special case

- The covariance between consumption growth and risky return is

\[ \text{cov}_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) = \text{cov}_t(\pi c_0 \epsilon_{t+1}, \pi r_0 \epsilon_{t+1}) = \pi c_0 \pi r_0 \sigma^2 \quad (165) \]

- So the risk premium can be written as

\[
\log \bar{R}_e - \log \bar{R}_f = -\pi_{\lambda,0} \pi_{r,0} \sigma^2 - \frac{1}{2} (\pi_{r0}^2 + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2) \sigma^2
\]

\[= \gamma \pi c_0 \pi r_0 \sigma^2 - \frac{1}{2} \left( \pi_{r0}^2 + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2 \right) \sigma^2
\]

\[= \gamma \text{cov}_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) - \frac{1}{2} \text{var}_t(r_{et+1})
\]
The role of habit

The risk premium is

\[
\log \bar{R}_e - \log \bar{R}_f = -\pi_{\lambda,0} \pi_{r,0} \sigma^2 - \frac{1}{2} (r_{r0}^2 + \sum_{j=0}^{\infty} (\pi_{\lambda,j} - \pi_{\lambda,j+1})^2) \sigma^2
\]

(167)

ignore the last term, we can see either we increase \(\pi_{\lambda,0}\) or \(\pi_{r,0}\).

In the case with habit

\[
\hat{\lambda}_t = -\frac{(1 + \beta h^2) \gamma}{(1 - \beta h)(1 - h)} \hat{c}_t + \frac{h \gamma}{(1 - \beta h)(1 - h)} \hat{c}_{t-1}
\]

\[+ \frac{\beta h \gamma}{(1 - \beta h)(1 - h)} E_t \hat{c}_{t+1}
\]

(168)

so we have

\[
\pi_{\lambda,0} = -\frac{(1 + \beta h^2) \gamma}{(1 - \beta h)(1 - h)} \pi_{c,0}
\]

(169)
The role of investment adjustment costs

- given $\pi_{c,0}$, the bigger is $h$, the bigger is $\pi_{\lambda,0}$
- However, in the production economy, $h$ would have a feedback effect. A big $h$ would imply more smoothed consumption. Namely it will reduce $\pi_{c,0}$. To prevent this to happen, we must add investment adjustment cost.
- investment adjustment cost would imply if big increases in investment is not desirable. And by

$$c_y \hat{c}_t + i_y t = \hat{y}_t$$

- an increase in output caused by technology shock, must be forced to consumption due to the above resource constraint.
The role of investment adjustment costs

- Remember

\[
R_{et+1} = \frac{q_{t+1} k_{t+2} + d_{t+1}}{q_t k_{t+1}}
\]

\[
= \frac{q_{t+1}[(1 - \delta)k_{t+1} + \varphi\left(\frac{i_{t+1}}{k_{t+1}}\right)k_{t+1}] + \alpha y_{t+1} - i_{t+1}}{q_t k_{t+1}}
\]

(171)

- and by

\[
1 = q_t \varphi'(\frac{i_t}{k_t})
\]

(172)

- We can have

\[
R_{et+1} = \frac{q_{t+1}[(1 - \delta) + \varphi\left(\frac{i_{t+1}}{k_{t+1}}\right)] + \alpha y_{t+1} - \varphi'(\frac{i_{t+1}}{k_{t+1}})\frac{i_{t+1}}{k_{t+1}}q_{t+1}}{q_t}
\]

(173)
The role of investment adjustment costs

- Linearization yields
  \[ \hat{R}_{et+1} = [1 - \beta(1 - \delta)](\hat{y}_{t+1} - \hat{k}_{t+1}) + \beta\hat{q}_{t+1} - \hat{q}_t \]

- where \( \hat{q}_t \)
  \[ \hat{q}_t = -\delta \varphi''(\delta)(\hat{i}_t - \hat{k}_t) \]

- so we have
  \[ \pi_{r,0} = (1 - \beta(1 - \delta))\pi_{y,0} + \beta\pi_{q,0} \]

- output
  \[ \hat{y}_t = A_t + \alpha\hat{k}_t + (1 - \alpha)\hat{n}_t \]

- if we fix labor in the steady-state, we have
  \[ \pi_{y,0} = 1 \]
The role of investment adjustment costs

so we have

$$\pi_{r,0} = (1 - \beta(1 - \delta)) + \beta \pi_{q,0}$$

and

$$\pi_{q,0} = -\delta \phi''(\delta) \pi_{i,0}$$

so to generate big risk premium we requires a big adjustment cost, namely we requires $\phi''(\delta)$ to be very negative given $\pi_{i,0}$.

However in the general equilibrium model, the adjustment cost in capital would smooth investment and hence make $\pi_{i,0}$ smaller. To prevent this, we need habit formation in consumption.