Notes on Asset Prices in the production economy-Qtheory

Pengfei Wang

Hong Kong University of Science and Technology

2010

Household

- A representative household with utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\log(c_t) - A_n \frac{n_t^{1+\gamma}}{1+\gamma}] \tag{1}$$

- The constraint:

$$c_t + V_t s_{t+1} + \frac{b_{t+1}}{R_{ft}} = (V_t + D_t) s_t + w_t n_t + b_t \tag{2}$$

- Notation: where c_t is consumption, w_t the really wage , n_t working hours , V_t the value of a standard share of the firm after dividend payment, and D_t is the dividend per share. The first order condition with respective to n_t and s_{t+1}, b_{t+1} are

Household

- The foc

$$\frac{1}{c_t} = \lambda_t \tag{3}$$

$$\lambda_t w_t = A_n n_t^{\gamma} \tag{4}$$

$$\frac{1}{R_{ft}} \lambda_t = \beta E_t \lambda_{t+1} \tag{5}$$

$$V_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [V_{t+1} + D_{t+1}] \tag{6}$$

- Transversaility conditions:

$$\lim_{j \to \infty} E_t \beta^j \lambda_{t+j} V_{t+j} s_{t+j+1} = 0 \tag{7}$$

$$\lim_{j \to \infty} E_t \beta^j \lambda_{t+j} \frac{b_{t+j+1}}{R_{ft+j}} = 0 \tag{8}$$

Household

- Iterating forward we obtain

$$V_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} + \lim_{j\to\infty} E_t \beta^j \frac{\lambda_{t+j}}{\lambda_t} V_{t+j} \tag{9}$$

- In equilibrium $s_{t+1} = 1$, $b_{t+1} = 0$, so we have

$$V_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \tag{10}$$

- the value of each share before the divident payment is

$$V_t + D_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \tag{11}$$

The firms

- The representive firm's problem is maximize the expected discounted divident or $(V_t + D_t)$

$$V_t + D_t = \max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \tag{12}$$

- The firm hires labor n_t from the household, accumulate capital, decides dividend payment with the following flow of fund constraint

$$D_t + k_{t+1} - (1-\delta)k_t = A_t k_t^{\alpha} n_t^{1-\alpha} - w_t n_t \tag{13}$$

- Set the bellman equation

$$W_t(k_t, A_t) = \max D_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} W_{t+1}(k_{t+1}, A_{t+1}) \tag{14}$$

The firms' foc

- The first order condition with respective to n_t, are

$$w_t n_t = (1-\alpha) y_t \tag{15}$$

- and

$$1 = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \frac{\partial W_{t+1}(k_{t+1}, A_{t+1})}{\partial k_{t+1}} \tag{16}$$

- Envelop theory:

$$\frac{\partial W_t(k_t, A_t)}{\partial k_t} = \alpha y_t / k_t + (1-\delta) \tag{17}$$

- so we have

$$\lambda_t = \beta E_t \lambda_{t+1} [\alpha y_{t+1} / k_{t+1} + (1-\delta)] \tag{18}$$

The firms' value after dividend payment

- We have obtained

$$\lambda_t = \beta E_t \lambda_{t+1} [\alpha y_{t+1} / k_{t+1} + (1 - \delta)] \quad (19)$$

- Multiply k_{t+1} to both side

$$\lambda_t k_{t+1} = \beta E_t \lambda_{t+1} [\alpha y_{t+1} + (1 - \delta) k_{t+1}] \quad (20)$$

- Use the law of capital

$$k_{t+2} = (1 - \delta) k_{t+1} + I_t \quad (21)$$

- we have

$$\begin{aligned} \lambda_t k_{t+1} &= \beta E_t \lambda_{t+1} [\alpha y_{t+1} - I_{t+1} + k_{t+2}] \\ &= \beta E_t \lambda_{t+1} [D_{t+1} + k_{t+2}] \end{aligned} \quad (22)$$

The firms' value after dividend payment

- This implies

$$k_{t+1} = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} = V_t \tag{23}$$

- Hence the value of firm before devident is just k_{t+1}

$$k_{t+1} + D_t = \alpha y_t + (1 - \delta) k_t \tag{24}$$

- Since

$$w_t = (1 - \alpha) A_t k_t^{\alpha} n_t^{-\alpha} \rightarrow n_t = \left(\frac{(1 - \alpha) A_t}{w_t} \right)^{\frac{1}{\alpha}} k_t \tag{25}$$

- so we have

$$y_t = A_t \left(\frac{(1 - \alpha) A_t}{w_t} \right)^{\frac{1-\alpha}{\alpha}} k_t \tag{26}$$

Then we have $\alpha y_t + (1 - \delta) k_t$ is proportional to firm's capital stock.

Equilibrium Condition

Equilibrium is defined in the usual way. Namely given the prices, household maximizes his utility and firm maximizes its objective function.

- The budget constraint:

$$c_t = D_t + w_t n_t = y_t - (k_{t+1} - (1-\delta)k_t) - w_t n_t + w_t n_t \tag{27}$$

- or we have

$$c_t + k_{t+1} = y_t + (1-\delta)k_t \tag{28}$$

- The asset value

$$\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}}[\alpha y_{t+1}/k_{t+1} + (1-\delta)] \tag{29}$$

- and the first order condition with labor

$$\frac{1}{c_t} w_t = A_n n_t^{\gamma} \tag{30}$$

- The production function

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha} \tag{31}$$

Equilibrium Condition

Verify the model have the same equilibrium as the following planner's problem:

- Utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [\log(c_t) - A_n \frac{n_t^{1+\gamma}}{1+\gamma}] \tag{32}$$

- Constraint:

$$c_t + k_{t+1} = y_t + (1-\delta) k_t \tag{33}$$

- production

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha} \tag{34}$$

Investment adjustment cost and the Q-theory

The household's problem is the same as before. The firm's problem is changed. We assume the firm need to pay investment adjustment cost.

- More specifically a firm needs to pay $I_t[1+\varphi(\frac{I_t}{k_t})]$ to increase capital by I_t units. The firm's objective function is still to maximize the max

$$\max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} \Leftrightarrow maxE_t \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} D_{t+j} \tag{35}$$

- The cash-flow of the firm changes to

$$D_t + I_t[1+\varphi(\frac{I_t}{k_t})] = A_t k_t^{\alpha} n_t^{1-\alpha} - w_t n_t \qquad (\lambda)$$

- And capital law of motion

$$k_{t+1} = (1-\delta)k_t + I_t \qquad (\mu)$$

Investment adjustment cost and the Q-theory

- The first order condition with respect to I_t, k_{t+1} yields

$$\lambda_t[1+\varphi(\frac{I_t}{k_t})+\frac{I_t}{k_t}\varphi'(\frac{I_t}{k_t})]=\mu_t \tag{36}$$

- and

$$\mu_t=\beta E_t\{(1-\delta)\mu_{t+1}+\lambda_{t+1}[\frac{\alpha y_{t+1}}{k_{t+1}}+\frac{I_{t+1}^2}{k_{t+1}^2}\varphi'(\frac{I_{t+1}}{k_{t+1}})]\} \tag{37}$$

- Define a variable $q_t=\frac{\mu_t}{\lambda_t}$, the above two equations can be expressed as

$$1+\varphi(\frac{I_t}{k_t})+\frac{I_t}{k_t}\varphi'(\frac{I_t}{k_t})=\frac{\mu_t}{\lambda_t}=q_t \tag{38}$$

- This is the marginal value of installed capital this period, which evolves according

$$q_t=\beta E_t\frac{\lambda_{t+1}}{\lambda_t}[(1-\delta)q_{t+1}+\frac{\alpha y_{t+1}}{k_{t+1}}+\frac{I_{t+1}^2}{k_{t+1}^2}\varphi'(\frac{I_{t+1}}{k_{t+1}})] \tag{39}$$

An example

- To get a better understanding of q_t, take a special form of $\varphi(\frac{I_t}{k_t}) = b\frac{I_t}{k_t}$, we then have

$$2b\frac{I_t}{k_t} = q_t - 1 \tag{40}$$

- or

$$I_t = \frac{(q_t - 1)}{2b} k_t \tag{41}$$

  we then have $I_t > 0$ if and only if $q_t > 1$.

- To increase one unit of installed capital, the firm needs to sacrifice $1 + 2b\frac{I_t}{k_t}$, so this is the marginal cost of installed capital. What is the gain? The cost-benefit analysis implies the marginal gain must equal to the marginal cost. So the gain would be q_t.

The value of firm in the example

- Using the equation for q_t, and the expression for I_t, we have

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[(1-\delta)q_{t+1} + \frac{\alpha y_{t+1}}{k_{t+1}} + \frac{I_{t+1}^2}{k_{t+1}^2}\varphi'(\frac{I_{t+1}}{k_{t+1}})] \tag{42}$$

- Mutiply both sides by k_{t+1} and recall $\varphi'(\frac{I_{t+1}}{k_{t+1}}) = b$

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[(1-\delta)q_{t+1}k_{t+1} + \alpha y_{t+1} + \frac{I_{t+1}^2}{k_{t+1}}b] \tag{43}$$

- to get obtain the firm value, recall

$$(1-\delta)k_{t+1} = k_{t+2} - I_{t+1} \tag{44}$$

The value of firm in the example

- to get obtain the firm value, recall

$$(1-\delta)k_{t+1} = k_{t+2} - I_{t+1} \tag{45}$$

- and

$$q_t = 2b\frac{I_t}{k_t} + 1 \tag{46}$$

- so we have Hence

$$\begin{aligned}(1-\delta)q_{t+1}k_{t+1} &= q_{t+1}k_{t+2} - q_{t+1}I_{t+1} \\ &= q_{t+1}k_{t+2} - (2b\frac{I_{t+1}}{k_{t+1}} + 1)I_{t+1}\end{aligned} \tag{47}$$

- And

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[(1-\delta)q_{t+1}k_{t+1} + \alpha y_{t+1} + \frac{I_{t+1}^2}{k_{t+1}}b]$$

The value of firm in the example

- We now have

$$\begin{aligned} q_t k_{t+1} &= \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[(1-\delta)q_{t+1}k_{t+1} + \alpha y_{t+1} + \frac{I_{t+1}^2}{k_{t+1}} b] \\ &= \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[q_{t+1}k_{t+2} + \alpha y_{t+1} - I_{t+1}(1 + b\frac{I_{t+1}}{k_{t+1}})] \\ &= \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[q_{t+1}k_{t+2} + D_{t+1}] \end{aligned} \tag{48}$$

- so we have

$$q_t k_{t+1} + D_t = V_t + D_t \tag{49}$$

Labor Decision

- Consider that the production function is

$$y_t = A_t F(k_t, n_t) \tag{50}$$

- where $F(k_t, n_t)$ exhibits constant return to scale. Given k_t, the decision for n_t is static.
- By CRS we can write

$$\max A_t F(k_t, n_t) - w_t n_t = \max_{\tilde{n}_t} [A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t] k_t \tag{51}$$

where $\tilde{n}_t = \frac{n_t}{k_t}$. the above equation yields

$$A_t F_2(1, \tilde{n}_t) = w_t \tag{52}$$

- This implies that $A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t$ would be a function of w_t and A_t, which does not depend on capital stock. We denote $A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t = R_t$

Cash flow

- Notice by
$$A_t F_2(1, \tilde{n}_t) = w_t \tag{53}$$
- the labor-capital ratio does not depends on firm's capital, k_t, it only depends on wage and technology level.
- Hence
$$R_t = A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t \tag{54}$$
is independent of firm's capital stock too.

Cash flow

- We now prove

$$R_t = AF'(K_t, N_t) \tag{55}$$

where K_t and N_t is the aggregate capital and labor. Notice in equilibrium $k_t = K_t, n_t = N_t$.

- first by CRS

$$A_t F(1, \frac{n_t}{k_t}) k_t = A_t F(k_t, n_t) \tag{56}$$

- so we have

$$w_t = A_t F_2(1, \frac{n_t}{k_t}) = A_t F_2(k_t, n_t) = A_t F_2(K_t, N_t) \tag{57}$$

- Take derivitative with respect to k_t we have

$$A_t F(1, \frac{n_t}{k_t}) - A_t F_2(1, \frac{n_t}{k_t}) \frac{n_t}{k_t} = A_t F_1(k_t, n_t) = A_t F_1(K_t, N_t) \tag{58}$$

- or

$$R_t = A_t F(1, \tilde{n}_t) - A_t F_2(1, \tilde{n}_t)\tilde{n}_t = A_t F(1, \tilde{n}_t) - w_t \tilde{n}_t = A_t F_1(K_t, N_t) \tag{59}$$

Cash flow

- with R_t, we have

$$d_t + I_t[1 + \varphi(\frac{I_t}{k_t})] = R_t k_t \tag{60}$$

The firm objective is

$$\begin{aligned} P_t &= max\ E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_{t+j} = D_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} P_{t+1} \\ &= \max\ E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} \left[R_t k_t - I_t[1 + \varphi(\frac{I_t}{k_t})] \right] \end{aligned}$$

s.t

$$k_{t+1} = (1 - \delta) k_t + I_t \tag{61}$$

FOCs

- define the investment rate as

$$I_t = i_t * k_t \tag{62}$$

- And guessing

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} \left[R_t k_t - I_t[1 + \varphi(\frac{I_t}{k_t})] \right] = p_t k_t \tag{63}$$

- by definition we then must have

$$\begin{aligned} p_t k_t &= \max[R_t k_t - I_t[1 + \varphi(\frac{I_t}{k_t})] + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} k_{t+1}] & (64) \\ &= \max[R_t k_t - I_t[1 + \varphi(\frac{I_t}{k_t})] + \beta(1 - \delta + i_t) k_t E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1}] & (65) \end{aligned}$$

FOCs

- or we have

$$p_t = \max[R_t - i_t[1 + \varphi(i_t)] + \beta(1 - \delta + i_t)E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1}] \tag{66}$$

- Define $E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} = q_t$, so we have

$$\varphi'(i_t)i_t + \varphi(i_t) = [q_t - 1] \tag{67}$$

- And value of firm after dividend is

$$\beta(1 - \delta + i_t)k_t E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} = q_t k_{t+1} \tag{68}$$

- By definition we have

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} \tag{69}$$

the firm's value

- Now we need to caculate p_t, by definition we have

$$p_t = R_t - i_t[1+\varphi(i_t)] + i_t q_t + (1-\delta)q_t \tag{70}$$

- since $R_t = A_t F_1(k_t, n_t)$ and $q_t = \varphi'(i_t)i_t + \varphi(i_t) + 1$, we have

$$\begin{aligned} p_t &= R_t + (1-\delta)q_t - i_t[1+\varphi(i_t)] + i_t q_t \\ &= R_t + (1-\delta)q_t - i_t[1+\varphi(i_t)] + \left[1+\varphi'(i_t)i_t + \varphi(i_t)\right] i_t \\ &= R_t + (1-\delta)q_t + \varphi'(i_t)i_t^2 \end{aligned} \tag{71}$$

- Hence we have

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[A_{t+1}F_1(k_{t+1}, n_{t+1}) + (1-\delta)q_{t+1} + \varphi'(\frac{I_{t+1}}{k_{t+1}})\frac{I_{t+1}^2}{k_{t+1}^2}] \tag{72}$$

the firm's value

- multiply both sides by k_{t+1} we have

$$\begin{aligned} q_t k_{t+1} &= \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[(1-\delta)q_{t+1}k_{t+1} + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1} \\ &\quad + \frac{I_{t+1}^2}{k_{t+1}}\varphi'(\frac{I_{t+1}}{k_{t+1}})] \end{aligned} \tag{73}$$

- using

$$q_{t+1} = 1 + \varphi'(\frac{I_{t+1}}{k_{t+1}})\frac{I_{t+1}}{k_{t+1}} + \varphi(\frac{I_{t+1}}{k_{t+1}}) \tag{74}$$

and

$$(1-\delta)k_{t+1} = k_{t+2} - I_{t+1} \tag{75}$$

- We can rewrite

$$
\begin{aligned}
&(1-\delta)q_{t+1}k_{t+1} + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1} + \frac{I_{t+1}^2}{k_{t+1}}\varphi'(\frac{I_{t+1}}{k_{t+1}}) \\
= \quad & q_{t+1}k_{t+2} - q_{t+1}I_t + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1} + \frac{I_{t+1}^2}{k_{t+1}}\varphi'(\frac{I_{t+1}}{k_{t+1}})
\end{aligned} \tag{76}
$$

- recall

$$
q_{t+1} = 1 + \varphi'(\frac{I_{t+1}}{k_{t+1}})\frac{I_{t+1}}{k_{t+1}} + \varphi(\frac{I_{t+1}}{k_{t+1}}) \tag{77}
$$

- so

$$
\begin{aligned}
&q_{t+1}k_{t+2} - q_{t+1}I_t + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1} + \frac{I_{t+1}^2}{k_{t+1}}\varphi'(\frac{I_{t+1}}{k_{t+1}}) \\
= \quad & q_{t+1}k_{t+2} - I_{t+1} - \varphi(\frac{I_{t+1}}{k_{t+1}})I_{t+1} + A_{t+1}F_1(k_{t+1}, n_{t+1})k_{t+1}
\end{aligned} \tag{78}
$$

the firm's value

- Recall

$$A_{t+1}F_1(k_{t+1},n_{t+1})k_{t+1} = A_t F(k_{t+1},n_{t+1}) - w_{t+1}n_{t+1} \tag{79}$$

$$D_{t+1} = A_t F(k_{t+1},n_{t+1}) - w_{t+1}n_{t+1} - I_{t+1} - \varphi(\frac{I_{t+1}}{k_{t+1}})I_{t+1} \tag{80}$$

- so we have

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[q_{t+1}k_{t+2} + D_{t+1}] \tag{81}$$

we established the proof.

different form of capital adjustment cost

- The firm's objective function is

$$\max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_t \tag{82}$$

- the dividend is

$$D_t = A_t F(k_t, n_t) - w_t n_t - I_t \tag{83}$$

- substitute the optimal labor choice as before

$$\max_{n_t} A_t F(k_t, n_t) - w_t n_t = R_t k_t \tag{84}$$

different form of capital adjustment cost

- And guessing the value function for the firm is linear

$$\max E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} D_t = p_t k_t \tag{85}$$

- by definition we have

$$\begin{aligned} p_t k_t &= \max[R_t k_t - I_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} k_{t+1}] && (86) \\ &= \max[R_t k_t - I_t + \beta(1 - \delta + \varphi(\frac{I_t}{k_t}))k_t E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1}] && (87) \end{aligned}$$

different form of capital adjustment cost

- as before define $\frac{I_t}{k_t} = i_t$, we have

$$p_t k_t = \max[R_t k_t - I_t + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} k_{t+1}] \quad (88)$$

$$= k_t \max_{i_t}[R_t - i_t + \beta(1 - \delta + \varphi(i_t)) E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1}] \quad (89)$$

- define $\beta E_t \frac{\lambda_{t+1}}{\lambda_t} p_{t+1} = q_t$. We have

$$1 = q_t \varphi'(i_t) \quad (90)$$

- hence we have

$$p_t = R_t - i_t + q_t(1 - \delta + \varphi(i_t)) \quad (91)$$

different form of capital adjustment cost

- To obtain q_t, we have

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[R_{t+1} - i_{t+1} + q_{t+1}(1 - \delta + \varphi(i_{t+1}))] \qquad (92)$$

- or

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[R_{t+1} k_{t+1} - I_{t+1} + q_{t+1}(1 - \delta + \varphi(\frac{I_{t+1}}{k_{t+1}}))k_{t+1}] \qquad (93)$$

- or we have

$$q_t k_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[D_{t+1} + q_{t+1} k_{t+2}] \qquad (94)$$

- so we have

$$V_t = q_t k_{t+1} \qquad (95)$$

Model setup-Bellman Equation

Now consider the centralized version with a special utility:
$u_t = \log(c_t - hc_{t-1}) - a_L n_t$

- There are $c_t, k_t, A_t,$, three state variables
- There are two constraints

$$c_t + i_t = y_t \tag{96}$$

$$k_{t+1} = (1-\delta)k_t + \varphi(\frac{i_t}{k_t})k_t \tag{97}$$

- Denote λ_t, χ_t are the Lagrangian mutlipliers of the above two constraints. Denote $q_t = \frac{\chi_t}{\lambda_t}$, we assume

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha}$$

Model setup-Bellman Equation

- So we set up the Bellman Equations as

$$
\begin{aligned}
V(c_{t-1}, k_t, A_t) &= \max_{c_t, n_t, i_t, k_{t+1}} \{ \frac{(c_t - hc_{t-1})^{1-\gamma}}{1-\gamma} - a_L n_t \\
&+ \beta E_t V(c_t, k_{t+1}, A_{t+1}) \\
&+ \lambda_t [A_t k_t^{\alpha} n_t^{1-\alpha} - c_t - i_t] \\
&+ \lambda_t q_t [(1-\delta) k_t + \varphi(\frac{i_t}{k_t}) k_t - k_{t+1}] \}
\end{aligned} \tag{98}
$$

Model setup-FOCs

- first order condtion with respect to c_t :

$$(c_t - hc_{t-1})^{-\gamma} + \beta E_t V_c'(c_t, k_{t+1}, A_{t+1}) = \lambda_t \tag{99}$$

- first order condtion with respect to n_t :

$$\lambda_t \frac{(1-\alpha)y_t}{n_t} = a_L \tag{100}$$

- first order condtion with respect to i_t :

$$\lambda_t = \lambda_t q_t \varphi'(\frac{i_t}{k_t}) \tag{101}$$

Model setup-FOCs

- first order condtion with respect to k_{t+1} :

$$\lambda_t q_t = \beta E_t V_k'(c_t, k_{t+1}, A_{t+1}) \tag{102}$$

- Envelop Theory

$$V_c^{'}(c_{t-1}, k_t, A_t) = -h(c_t - hc_{t-1})^{-\gamma} \tag{103}$$

$$V_k^{'}(c_{t-1}, k_t, A_t) = \lambda_t \frac{\alpha y_t}{k_t} + \lambda_t q_t[(1-\delta) + \varphi(\frac{i_t}{k_t}) - \varphi'(\frac{i_t}{k_t})\frac{i_t}{k_t}] \tag{104}$$

Model setup-FOCs

- So the first order conditions are be summarized by

$$\lambda_t = (c_t - hc_{t-1})^{-\gamma} - \beta h E_t (c_{t+1} - hc_t)^{-\gamma} \tag{105}$$

$$\lambda_t \frac{(1-\alpha) y_t}{n_t} = a_L \tag{106}$$

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \{ \frac{\alpha y_{t+1}}{k_{t+1}} + q_{t+1} [(1-\delta) + \varphi(\frac{i_{t+1}}{k_{t+1}}) - \varphi'(\frac{i_{t+1}}{k_{t+1}}) \frac{i_{t+1}}{k_{t+1}}] \} \tag{107}$$

$$1 = q_t \varphi'(\frac{i_t}{k_t}) \tag{108}$$

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha} = c_t + i_t \tag{109}$$

- and capital follows

$$k_{t+1} = (1-\delta) k_t + \varphi(\frac{i_t}{k_t}) k_t \tag{110}$$

Equivalence

- The only difference between the decentralized version and plan'er problem is equation regarding q_t

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}\{\frac{\alpha y_{t+1}}{k_{t+1}} + q_{t+1}[(1-\delta) + \varphi(\frac{i_{t+1}}{k_{t+1}}) - \varphi'(\frac{i_{t+1}}{k_{t+1}})\frac{i_{t+1}}{k_{t+1}}]\} \tag{111}$$

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}[R_{t+1} - i_{t+1} + q_{t+1}(1-\delta+\varphi(i_{t+1}))] \tag{112}$$

- We now prove these two are the same. Notice

$$1 = q_t \varphi'(\frac{i_t}{k_t}) \tag{113}$$

- so we have

$$-i_{t+1} = -i_{t+1} q_{t+1} \varphi'(\frac{i_{t+1}}{k_{t+1}}) \tag{114}$$

Asset price implication

- Recall the risk free rate is

$$\lambda_t = \beta R_{ft} E_t \lambda_{t+1} \tag{115}$$

- and the stock price

$$\lambda_t V_t = \beta E_t \lambda_{t+1} [V_{t+1} + D_{t+1}] \tag{116}$$

- or

$$\lambda_t = \beta E_t \lambda_{t+1} [\frac{V_{t+1} + D_{t+1}}{V_t}] = \beta E_t \lambda_{t+1} R_{et+1} \tag{117}$$

Asset price implication

- The log linearize return

$$1 = \beta E_t \exp(\log \lambda_{t+1} - \log \lambda_t + r_{ft}) \tag{118}$$

- and the stock price

$$1 = \beta E_t[\log \lambda_{t+1} - \log \lambda_t + \log(V_{t+1} + D_{t+1}) - \log V_t] \tag{119}$$

- or

$$1 = \beta E_t \exp(\log \lambda_{t+1} - \log \lambda_t + r_{et+1}) \tag{120}$$

- we are interested in the risk premium

$$\Delta r = E_t r_{et+1} - r_{ft} \tag{121}$$

Direct log-linearization

- The log-linearization around the deterministic steady-state. In the deterministic steady-state, there is no uncertainty we must have

$$r_f = r_e = -\log(\beta) \tag{122}$$

- so the above two equations implies

$$E_t\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_{ft} = 0 \tag{123}$$

and

$$E_t\hat{\lambda}_{t+1} - \hat{\lambda}_t + E_t\hat{r}_{et+1} = 0 \tag{124}$$

- The premium is

$$\begin{aligned} E_t r_{et+1} - r_{ft} &= E_t r_{et+1} - r_e + r_e - r_{ft} \\ &= E_t \hat{r}_{et+1} - \hat{r}_{ft} \\ &= 0 \end{aligned}$$

Indirect log-linearization

- We have see that directly using log-linearization generate zero risk premium. So we need to look at some other methods.
- We now look at the second order approximation. For R_{ft}, define $\bar{R}_f = ER_{ft}$ as the unconditional mean of risk free rate rate and $\bar{R}_e = ER_{et}$ similary. We have

$$
\begin{aligned}
R_{ft} &= \frac{1}{\beta E_t \exp(\log \lambda_{t+1} - \log \lambda_t)} && (125) \\
&= \frac{1}{\beta E_t \exp(\log \lambda_{t+1} - \log \lambda - (\log \lambda_t - \log \lambda))} \\
&= \frac{1}{\beta} E_t \exp(-\hat{\lambda}_{t+1} + \hat{\lambda}_t)
\end{aligned}
$$

Indirect log-linearization

- for real variable we have

$$\hat{x}_t = \pi^x_k \hat{k}_t + \pi^x_A \hat{A}_t \tag{126}$$

$$\hat{k}_{t+1} = \pi^k_k \hat{k}_t + \pi^k_A \hat{A}_t$$

- where

$$\hat{A}_t = \rho \hat{A}_t + \varepsilon_t \tag{127}$$

- where ε_t is a normal distribution with standard deviation σ. Iteration we can write

$$\hat{k}_{t+1} = \frac{\pi^k_A \hat{A}_t}{1 - \pi^k_k L} = \frac{\pi^k_A}{1 - \pi^k_k L} \frac{\varepsilon_t}{1 - \rho L} \tag{128}$$

- so

$$\hat{x}_t = \frac{\pi^x_k \pi^k_A}{1 - \pi^k_k L} \frac{L \varepsilon_t}{1 - \rho L} + \frac{\pi^x_A \varepsilon_t}{1 - \rho L}$$

Indirect log-linearization

- We have

$$\hat{x}_t = \frac{\pi_k^x \pi_A^k}{1 - \pi_k^k L} \frac{L\varepsilon_t}{1 - \rho L} + \frac{\pi_A^x \varepsilon_t}{1 - \rho L} \tag{129}$$

- It implis we can write

$$\hat{x}_t = \sum_{j=0}^{\infty} \pi_j^x \varepsilon_{t-j} \tag{130}$$

- so the log difference between a variable x_t and its deterministic steady-state are normally distributed.

Indirect log-linearization

- Using the above fact we have

$$\hat{\lambda}_t = \sum_{j=0}^{\infty} \pi_{\lambda,j} \varepsilon_{t-j} \tag{131}$$

- and

$$\hat{\lambda}_{t+1} = \sum_{j=0}^{\infty} \pi_{\lambda,j} \varepsilon_{t+1-j} \tag{132}$$

- The expected value

$$E_t \hat{\lambda}_{t+1} = \sum_{j=0}^{\infty} \pi_{\lambda,j+1} \varepsilon_{t-j} \tag{133}$$

- Conditional variance

$$Var_t(\hat{\lambda}_{t+1}) = \pi_{\lambda,0}^2 \sigma^2 \tag{134}$$

Indirect log-linearization

- So the implied risk free rate is

$$
\begin{aligned}
R_{ft} &= \frac{1}{\beta}\frac{1}{E_t \exp\{\hat{\lambda}_t - \hat{\lambda}_{t+1}\}} \\
&= \frac{1}{\beta}\exp\{-E_t\hat{\lambda}_{t+1} + \hat{\lambda}_t - \frac{1}{2}var_t(\hat{\lambda}_{t+1})\} \qquad (135)\\
&= \frac{1}{\beta}\exp\{\sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})\varepsilon_{t-j} - \frac{1}{2}\pi_{\lambda,0}^2\sigma^2\} \qquad (136)
\end{aligned}
$$

- we take log and we then have

$$
r_{ft} = -\log\beta + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})\varepsilon_{t-j} - \frac{1}{2}\pi_{\lambda,0}^2\sigma^2 \qquad (137)
$$

- so the unconditional mean is

$$
Er_{ft} = -\log\beta - \frac{1}{2}\pi_{\lambda,0}^2\sigma^2 \qquad (138)
$$

- which is different from the deterministic steady-state.

Indirect log-linearization

- To obtain the unconditional mean of R_{ft} we use

$$R_{ft} = \frac{1}{\beta}\exp\{\sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})\varepsilon_{t-j} - \frac{1}{2}\pi_{\lambda,0}^2\sigma^2\} \tag{139}$$

- Again $\sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})\varepsilon_{t-j}$ has normal distribution we have

$$ER_{ft} = \bar{R}_f = \frac{1}{\beta}\exp(-\frac{1}{2}\pi_{\lambda,0}^2\sigma^2 + \frac{1}{2}\sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2\sigma^2) \tag{140}$$

- The volatility of risk free rate

$$r_{ft} = -\log\beta + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})\varepsilon_{t-j} + \frac{1}{2}\pi_{\lambda,0}^2\sigma^2 \tag{141}$$

- so we have

$$var(r_{ft}) = \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2\sigma^2 \tag{142}$$

Indirect log-linearization of risky return

- The risk return

$$1 = \beta E_t \exp\{\hat{\lambda}_{t+1} - \hat{\lambda}_t + r_{et+1}\} \tag{143}$$

- Define $\bar{R}_e = E_t R_{et}$

$$\begin{aligned} 1 &= \beta E_t \exp\{\hat{\lambda}_{t+1} - \hat{\lambda}_t + r_{et+1}\} \\ &= \beta \bar{R}_e E_t \exp\{\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_{et+1}\} \end{aligned} \tag{144}$$

Indirect log-linearization of risky return

- Using the property of normal distribution , the term $\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_{et+1}$ is also normally distributed.
- Its mean

$$E_t[\hat{\lambda}_{t+1} - \hat{\lambda}_t] = \sum(\pi_{\lambda,j+1} - \pi_{\lambda,j} + \pi_{r,j})\varepsilon_{t-j} \tag{145}$$

- its variance is

$$\begin{aligned} Var_t(\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{r}_{et+1}) &= Var_t(\pi_{\lambda,0}\varepsilon_{t+1} + \pi_{r,0}\varepsilon_{t+1}) \\ &= (\pi^2_{\lambda,0} + \pi^2_{r0} + 2\pi_{\lambda,0}\pi_{r,0})\sigma^2 \end{aligned} \tag{146}$$

- so we have

$$\bar{R}_e = \frac{1}{\beta}\exp\left[\begin{array}{c} \sum(-\pi_{\lambda,j+1} + \pi_{\lambda,j} - \pi_{r,j+1})\varepsilon_{t-j} \\ -\frac{1}{2}(\pi^2_{\lambda,0} + \pi^2_{r0} + 2\pi_{\lambda,0}\pi_{r,0})\sigma^2 \end{array}\right] \tag{147}$$

Indirect log-linearization of risky return

- or we have

$$\log \bar{R}_e = -\log \beta - \sum(\pi_{\lambda,j+1} - \pi_{\lambda,j} + \pi_{r,j+1})\varepsilon_{t-j} - \frac{1}{2}(\pi_{\lambda,0}^2 + \pi_{r0}^2 + 2\pi_{\lambda} \tag{148}$$

- so we must have

$$\pi_{\lambda,j+1} - \pi_{\lambda,j} + \pi_{r,j+1} = 0 \tag{149}$$

- or we have

$$\log \bar{R}_e = -\log \beta - \frac{1}{2}(\pi_{\lambda,0}^2 + \pi_{r0}^2 + 2\pi_{\lambda,0}\pi_{r,0})\sigma^2 \tag{150}$$

- and the risk rate is

$$\log \bar{R}_f = -\log \beta + \frac{1}{2}\sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2\sigma^2 - \frac{1}{2}\pi_{\lambda,0}^2\sigma^2 \tag{151}$$

Indirect log-linearization of risky return

- The premium

$$\log \bar{R}_e = -\log \beta - \frac{1}{2}(\pi_{\lambda,0}^2 + \pi_{r0}^2 + 2\pi_{\lambda,0}\pi_{r,0})\sigma^2 \tag{152}$$

$$\log \bar{R}_f = -\log \beta - \frac{1}{2}\pi_{\lambda,0}^2\sigma^2 + \frac{1}{2}\sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2\sigma^2 \tag{153}$$

- or we have

$$\log \bar{R}_e - \log \bar{R}_f = -\pi_{\lambda,0}\pi_{r,0}\sigma^2 - \frac{1}{2}(r_{r0}^2 + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2)\sigma^2 \tag{154}$$

Indirect log-linearization of risky return

- we have

$$\log \bar{R}_e - \log \bar{R}_f = -\pi_{\lambda,0}\pi_{r,0}\sigma^2 - \frac{1}{2}(\pi_{r0}^2 + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2)\sigma^2 \tag{155}$$

- Finally by

$$\pi_{\lambda,j+1} - \pi_{\lambda,j} + \pi_{r,j+1} = 0 \tag{156}$$

- we can write

$$\pi_{r0}^2 + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2 = \pi_{r0}^2 + \sum_{j=0}^{\infty}\pi_{r,j+1}^2 = \sum_{j=0}^{\infty}\pi_{r,j}^2\sigma^2 \tag{157}$$

- The unconditinal volatility of

$$var(\hat{r}_{et}) = var(r_{et}) = var(\sum_{j=0}^{\infty}\pi_{r,j}\varepsilon_{t-j}) = \sum_{j=0}^{\infty}\pi_{r,j}^2\sigma^2 \tag{158}$$

A speical case

- Consider a special case, with $h = 0$. So we have

$$\hat{\lambda}_t = -\gamma \hat{c}_t \tag{159}$$

- so we have

$$\hat{\lambda}_t = -\gamma \sum \pi_{c,j} \varepsilon_{t-j} \tag{160}$$

- the consumption growth is

$$\hat{c}_{t+1} - \hat{c}_t = \pi_{c0} \varepsilon_{t+1} + \sum_{j=0}^{\infty} (\pi_{c,j+1} - \pi_{\lambda,j}) \varepsilon_{t-j} \tag{161}$$

- The covariance between consumption growth and risky return is

$$cov_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) = cov_t(\pi_{c0} \varepsilon_{t+1}, \pi_{r0} \varepsilon_{t+1}) = \pi_{c0} \pi_{r0} \sigma^2 \tag{162}$$

A speical case

- The covariance between consumption growth and risky return is

$$cov_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) = cov_t(\pi_{c0}\varepsilon_{t+1}, \pi_{r0}\varepsilon_{t+1}) = \pi_{c0}\pi_{r0}\sigma^2 \quad (163)$$

- So the risk premium can be written as

$$\begin{aligned} \log \bar{R}_e - \log \bar{R}_f &= -\pi_{\lambda,0}\pi_{r,0}\sigma^2 - \frac{1}{2}(\pi_{r0}^2 + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2)\sigma^2 \quad (164) \\ &= \gamma\pi_{c0}\pi_{r0}\sigma^2 - \frac{1}{2}(\pi_{r0}^2 + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2)\sigma^2 \\ &= \gamma cov_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) - \frac{1}{2}var_t(r_{et+1}) \end{aligned}$$

A special case

- The covariance between consumption growth and risky return is

$$cov_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) = cov_t(\pi_{c0}\varepsilon_{t+1}, \pi_{r0}\varepsilon_{t+1}) = \pi_{c0}\pi_{r0}\sigma^2 \quad (165)$$

- So the risk premium can be written as

$$\begin{aligned} \log \bar{R}_e - \log \bar{R}_f &= -\pi_{\lambda,0}\pi_{r,0}\sigma^2 - \frac{1}{2}(\pi_{r0}^2 + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2)\sigma^2 \quad (166) \\ &= \gamma\pi_{c0}\pi_{r0}\sigma^2 - \frac{1}{2}(\pi_{r0}^2 + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2)\sigma^2 \\ &= \gamma cov_t(\hat{c}_{t+1} - \hat{c}_t, \hat{r}_{et+1}) - \frac{1}{2}var_t(r_{et+1}) \end{aligned}$$

The role of habit

- The risk premium is

$$\log \bar{R}_e - \log \bar{R}_f = -\pi_{\lambda,0}\pi_{r,0}\sigma^2 - \frac{1}{2}(r_{r0}^2 + \sum_{j=0}^{\infty}(\pi_{\lambda,j} - \pi_{\lambda,j+1})^2)\sigma^2 \tag{167}$$

- ignore the last term, we can see either we increase $\pi_{\lambda,0}$ or π_{r0}.
- In the case with habit

$$\hat{\lambda}_t = -\frac{(1+\beta h^2)\gamma}{(1-\beta h)(1-h)}\hat{c}_t + \frac{h\gamma}{(1-\beta h)(1-h)}\hat{c}_{t-1} + \frac{\beta h\gamma}{(1-\beta h)(1-h)}E_t\hat{c}_{t+1} \tag{168}$$

- so we have

$$\pi_{\lambda,0} = -\frac{(1+\beta h^2)\gamma}{(1-\beta h)(1-h)}\pi_{c,0} \tag{169}$$

The role of investment adjustment costs

- given $\pi_{c,0}$, the biger is h, the biger is $\pi_{\lambda,0}$
- However, in the production economy, h would have a feedback effect. A big h would implies more smoothed consumption. Namely it will reduce $\pi_{c,0}$. To prevent this to happen, we must add investment adjustment cost.
- investment adjustment cost would implies if big increases in investment is not desirable. And by

$$c_y \hat{c}_t + i_{y\,t} = \hat{y}_t \tag{170}$$

- an increase in output cuased by technology shock, must be forced to consumption due to the above resource constraint.

- Remember

$$R_{et+1} = \frac{q_{t+1}k_{t+2}+d_{t+1}}{q_t k_{t+1}} = \frac{q_{t+1}[(1-\delta)k_{t+1}+\varphi(\frac{i_{t+1}}{k_{t+1}})k_{t+1})+\alpha y_{t+1}-i_{t+1}}{q_t k_{t+1}} \quad (171)$$

- and by

$$1 = q_t\varphi'(\frac{i_t}{k_t}) \quad (172)$$

- We can have

$$R_{et+1} = \frac{q_{t+1}[(1-\delta)+\varphi(\frac{i_{t+1}}{k_{t+1}}))+\alpha\frac{y_{t+1}}{k_{t+1}}-\varphi'(\frac{i_{t+1}}{k_{t+1}})\frac{i_{t+1}}{k_{t+1}}q_{t+1}}{q_t} \quad (173)$$

The role of investment adjustment costs

- Linearization yields

$$\hat{R}_{et+1} = [1-\beta(1-\delta)][\hat{y}_{t+1} - \hat{k}_{t+1}] + \beta\hat{q}_{t+1} - \hat{q}_t$$

- where $\hat{q}_t$

$$\hat{q}_t = -\delta\varphi''(\delta)[\hat{\imath}_t - \hat{k}_t] \tag{174}$$

- so we have

$$\pi_{r,0} = (1-\beta(1-\delta))\pi_{y,0} + \beta\pi_{q,0} \tag{175}$$

- output

$$\hat{y}_t = A_t + \alpha\hat{k}_t + (1-\alpha)\hat{n}_t \tag{176}$$

- if we fix labor in the steady-state, we have

$$\pi_{y,0} = 1 \tag{177}$$

The role of investment adjustment costs

- so we have

$$\pi_{r,0} = (1 - \beta(1 - \delta)) + \beta\pi_{q,0} \tag{178}$$

- and

$$\pi_{q,0} = -\delta\varphi''(\delta)\pi_{i,0} \tag{179}$$

- so to generate big risk premium we requires a big adjustment cost, namely we requires $\varphi''(\delta)$ to be very negative given $\pi_{i,0}$.
- However in the general equilibrium model, the adjustment cost in capital would smooth investment and hence make $\pi_{i,0}$ smaller. To provent this, we need habit formation in consumption.