Introduction

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Introduction

On the surface, the history of macroeconomics in the twentieth century appears as a series of battles, revolutions, and counterrevolutions

- Keynesian revolution of the 1930s and 1940s
- Monetarists and Keynesians of the 1950s and 1960s
- The Rational Expectations revolution of the 1970s
- New Keynesians and New Classicals of the 1980s
- Model with frictions, credit cycle, Labor search, imperfect information of 1990s-

Key feature of macroeconomics

- Dynamics
- Uncertainty
- General Equilibrium

Key feature of macroeconomics

Modern Macroeconomics reply on Models:

- Explicit microeconomic foundation
- Robust to Lucas critique
- Policy Analysis
- Forecasting
- Counterfactuals

Some useful example

Consider a two-period problem in which an individual decide how to spend his wealth s_0 in two periods. He enjoys utility

$$u(c_1, c_2) = \ln(c_1) + \ln(c_2) \tag{1}$$

this person face constraints

$$c_1 + s_1 = s_0 \tag{2}$$

and

$$c_2 = s_1 \tag{3}$$

Some useful example

We set up the Lagrangian function to solve this problem

$$\mathcal{L} = \ln(c_1) + \ln(c_2) + \lambda_1[s_0 - s_1 - c_1] + \lambda_2[s_1 - c_2] \tag{4}$$

the f.o.c conditions

$$\frac{1}{c_1} = \lambda_1 \tag{5}$$

$$\frac{1}{c_2} = \lambda_2 \tag{6}$$

$$\lambda_1 = \lambda_2 \tag{7}$$

we can solve this problem easily

$$c_1 = c_2 = \frac{1}{2}s_0; s_1 = \frac{1}{2}s_0 \tag{8}$$

and we have

$$\mathcal{L} = 2\log(\frac{1}{2}) + 2\ln s_0 \tag{9}$$

Some useful example (continued)

Now we extend this example to a finite T periods

$$u(c_1, c_2, ...c_T) = \ln(c_1) + \ln(c_2) + ... + \ln(c_T)$$

with the constraint

$$\begin{aligned} c_1 + s_1 &= s_0 \\ c_2 + s_2 &= s_1 \\ &\ldots \\ c_T + s_T &= s_{T-1} \end{aligned} \tag{10}$$

and $s_T \geq 0$

Similarly we can set up the Lagrangian function as

$$\begin{aligned} \mathcal{L} = &\ln(c_1) + \ln(c_2) + ... + \ln(c_T) \\ &+\lambda_1[s_0 - s_1 - c_1] + \lambda_2[s_1 - c_2 - s_2] \\ &+.... \\ &+\lambda_T[s_{T-1} - c_T - s_T] + \mu s_T \end{aligned} \tag{11}$$

Some useful example (continued)

- f.o.c with respect to $\{c_1, c_2, ..., c_T\}$

$$\begin{aligned} \frac{1}{c_1} &= \lambda_1 \\ \frac{1}{c_2} &= \lambda_2 \\ &\cdots \\ \frac{1}{c_T} &= \lambda_T \end{aligned} \tag{12}$$

Some useful example (continued)

- f.o.c with respect to $\{s_1, s_2, ..., s_{T-1}\}$

$$
\begin{aligned}
-\lambda_1 + \lambda_2 &= 0 \\
-\lambda_2 + \lambda_3 &= 0 \\
&\dots \\
-\lambda_{T-1} + \lambda_T &= 0
\end{aligned} \tag{13}
$$

- f.o.c with respect to s_T

$$
-\lambda_T + \mu = 0 \tag{14}
$$

Some useful example (continued)

- $\{\lambda_t\} > 0$ for $t \geq 1$,so we have $\mu = \lambda_T > 0$ and we must have $s_T = 0$.
- $\lambda_1 = \lambda_2 = ... = \lambda_T$ implies $c_1 = c_2 = ... = c_T$
- the constraint can simplified into

$$c_1 + c_2 + ...c_T = s_0 \tag{15}$$

- hence we have

$$c_t^* = \frac{1}{T} s_0 \tag{16}$$

define the problem recursively

$$V_1(s_0) = \max\{\ln(c_1) + \ln(c_2) + ... + \ln(c_T)\}$$

with the constraint (10) and by the above caculation we have

$$V_1(s_0) = T\ln(\frac{s_0}{T})$$

- given s_1, we define a $T-1$ period maximizaiton problem as

$$V_2(s_1) = \max\{\ln(c_2) + \ln(c_3) + ... + \ln(c_T)\} \tag{17}$$

with

$$\begin{aligned} c_2 + s_2 &= s_1 \\ c_3 + s_3 &= s_2 \\ &\dots \\ c_T + s_T &= s_{T-1} \end{aligned} \tag{18}$$

and $s_T \geq 0$

define the problem recursively (continued)

- By similarly method we can have

$$V_2(s_1) = (T-1)\ln(\frac{s_1}{T-1}) \tag{19}$$

- Now solving a two-period problem

$$W(s_0) = \max\{\ln(s_0 - s_1) + (T-1)\ln(\frac{s_1}{T-1})\}$$

- f.o.c

$$\frac{1}{s_0 - s_1} = \frac{T-1}{s_1} \tag{20}$$

define the problem recursively (continued)

- $\frac{1}{s_0 - s_1} = \frac{T-1}{s_1}$ leads to

$$s_1 = \frac{T-1}{T} s_0,\ c_1 = \frac{1}{T} s_0 \tag{21}$$

- and the value function $W(s_0) = T \ln(\frac{s_0}{T}) = V_1(s_0)$.
- by construction, it suggests

$$V_1(s_0) = \max_{c_1, s_1} \{\ln(c_1) + V_2(s_1)\} \tag{22}$$

with

$$c_1 + s_1 = s_0 \tag{23}$$

define the problem recursively (continued)

- similary given s_2, we define a $T-2$ period maximizaiton problem as

$$V_3(s_2) = \max\{\ln(c_3).... + \ln(c_T)\} \tag{24}$$

with

$$\begin{aligned} c_3 + s_3 &= s_2 && (25) \\ c_4 + s_4 &= s_3 && (26) \\ &\dots \\ c_T + s_T &= s_{T-1} \end{aligned}$$

and $s_T \geq 0$

- verify

$$V_2(s_1) = \max_{c_2, s_2}\{\ln(c_2) + V_3(s_2)\}$$

Extending it to infinite

- Now we extend the model to infinite period. This person maximize

$$\max \sum_{t=1}^{\infty} \beta^{t-1} \log(c_t) \tag{27}$$

with the sequence of infinite constraint

$$c_t + s_{t+1} = s_t \tag{28}$$

Define

$$V(s_1) = \max \sum_{t=1}^{\infty} \beta^{t-1} \log(c_t) \tag{29}$$

As the value of this ininfite sum of utility level.

Solving the infinite problem

- Set the Lagrangian function as

$$\mathcal{L} = \lim_{T \to \infty} \sum_{t=1}^{T} \beta^{t-1} \left[\log(c_t) + \lambda_t (s_t - s_{t+1} - c_t) \right] \tag{30}$$

- f.o.c with respect to c_t

$$\frac{1}{c_t} = \lambda_t \tag{31}$$

- foc with respect to s_{t+1}, for $t = 1, 2, ...,$

$$-\beta^{t-1} \lambda_t + \beta^t \lambda_{t+1} = 0 \tag{32}$$

Solving the infinite problem

- and for S_{T+1}, we have

$$\frac{\partial \mathcal{L}}{\partial S_{T+1}} = -\beta^T \lambda_T \leq 0$$

we either have $s_{T+1} = 0$, and if $s_{T+1} > 0$, we must have $-\beta^T \lambda_t = 0$, we can write it as

$$\beta^T \lambda_T s_{T+1} = 0 \tag{33}$$

Since it holds for any T, we then have

$$\lim_{T \to \infty} \beta^T \lambda_T s_{T+1} = 0 \tag{34}$$

This is called Transversality condition.

Solving the infinite problem (continued)

- $$-\beta^{t-1}\lambda_t + \beta^t \lambda_{t+1} = 0 \tag{35}$$

  implies

  $$-\beta^{t-1}\frac{1}{c_t} + \beta^t \frac{1}{c_{t+1}} = 0 \tag{36}$$

  or

  $$c_{t+1} = \beta c_t \tag{37}$$

- Given any c_1, we have $\beta^T \lambda_T s_{T+1} = s_{T+1}\beta^T \frac{1}{(\beta)^{T-1}c_1} = \beta \frac{S_{T+1}}{c_1}$, so transversality implies

  $$\lim_{T\to\infty} \beta^T \lambda_T s_{T+1} = 0 = \beta \lim_{T\to\infty} \frac{S_{T+1}}{c_1} = 0 \tag{38}$$

  or

  $$\lim_{T\to\infty} S_{T+1} = 0$$

Solving the infinite problem (continued)

- Finally use the budget constraint

$$\begin{aligned} c_1 + s_2 &= s_1 \\ c_2 + s_3 &= s_2 \\ c_3 + s_4 &= s_3 \\ &\ldots \\ c_T + s_{T+1} &= s_T \\ &\ldots \end{aligned} \tag{39}$$

suming these we have

$$c_1 + c_2 + ... + \lim_{T \to \infty} s_{T+1} = s_1 \tag{40}$$

Solving the infinite problem (continued)

- It follows

$$c_1 = (1-\beta)s_1 \tag{41}$$

$$s_2 = \beta s_1 \tag{42}$$

and in general in each period, we have

$$c_t = (1-\beta)s_t \tag{43}$$

$$s_{t+1} = \beta s_t \tag{44}$$

In this problem, these two functions are called as policy functions.

Solving the infinite problem recursively

- Now re-solved this problem recursively. In period t, starting with s_t the agent solves

$$V(s_t) = \max\{\log(c_t) + \beta\log(c_{t+1}) + \beta^2\log(c_{t+2}) + ..\}$$
$$= \max\sum_{j=0}^{\infty}\beta^j\log(c_{t+j})$$

with the period-by-period constraint

$$c_{t+j} + s_{t+j+1} = s_{t+j} \tag{45}$$

for $j = 0, 1, 2, ...,$

- The solution is the same as before, we have $c_{t+j} = (1-\beta)s_{t+j}$ and $s_{t+1+j} = \beta s_{t+j}$ for $j \geq 0$.

Solving the infinite problem recursively

- Using the policy function we obtain

$$V(s_t) = \frac{\log(1-\beta) + \log(s_t)}{1-\beta} + \frac{\log(\beta)\beta}{(1-\beta)^2} \tag{46}$$

  we call $V(s_t)$ as value function. We now show, the above can solved recursively by value function.

- Now verify

$$V(s_t) = \max_{s_{t+1}, c_t} \{\log(c_t) + \beta V(s_{t+1})\} \tag{47}$$

  with

$$c_t + s_{t+1} = s_t \tag{48}$$

  equation (47) is called as Bellman equation.

Solving the infinite problem recursively (continued)

- Plug the value function in

$$V(s_t) = \max_{s_{t+1}, c_t} \left\{ \log(c_t) + \beta \left[\frac{\log(1-\beta) + \log(s_{t+1})}{1-\beta} + \frac{\log(\beta)\beta}{(1-\beta)^2} \right] \right\} \tag{49}$$

with

$$c_t + s_{t+1} = s_t \tag{50}$$

the solution can be founded by

$$\frac{1}{s_t - s_{t+1}} = \beta \frac{1-\beta}{s_{t+1}} \tag{51}$$

and we now have

$$V(s_t) = \frac{\log(1-\beta) + \log(s_t)}{1-\beta} + \frac{\log(\beta)\beta}{(1-\beta)^2}$$

Solving the infinite problem recursively (continued)

- by

$$s_{t+1} = \beta s_t; c_t = (1-\beta)s_t \tag{52}$$

and we now have

$$\begin{aligned} V(s_t) &= \log(1-\beta) + \log(s_t) \\ &\quad + \beta\left[\frac{\log(1-\beta)+\log(s_t)+\log(\beta)}{1-\beta} + \frac{\log(\beta)\beta}{(1-\beta)^2}\right] \\ &= \frac{\log(1-\beta)+\log(s_t)}{1-\beta} + \frac{\log(\beta)\beta}{(1-\beta)^2} \end{aligned}$$

- so indeed we have

$$V(s_t) = \max\{\log(c_t) + \beta V(s_{t+1})\} \tag{53}$$

Solving A general infinite problem recursively

- Let $\beta \in (0,1)$ be a discount factor, we want to choose an infinite sequence of "control", $\{u_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

subject to

$$x_{t+1} = g(x_t, u_t) \tag{54}$$

- dynamic programing seeks a time-invariant policy function , mapping *state* x_t to control u_t, such that

$$u_t = h(x_t) \tag{55}$$

$$x_{t+1} = g(x_t, u_t) \tag{56}$$

starting from initial condition x_0.

Solving A general infinite problem recursively

- A recursive formation of the problem is to define a value function V, such that

$$
\begin{aligned}
V(x_t) &= \max \sum_{j=0}^{\infty} \beta^j r(x_{t+j}, u_{t+j}) \qquad (57\\
&= \max\{r(x_t, u_t) + \beta[r(x_{t+1}, u_{t+1}) + \beta r(x_{t+2}, u_{t+2}) + ...]\}\\
&= \max\{r(x_t, u_t) + \max \beta[r(x_{t+1}, u_{t+1}) + \beta r(x_{t+2}, u_{t+2}) + ...]\\
&= \max\{r(x_t, u_t) + \beta V(x_{t+1})\}
\end{aligned}
$$

Solving A general infinite problem recursively

- Proof. Suppose $\{x^*_{t+j+1}, u^*_{t+j}\}_{j=0}^{\infty}$ are the solution , so we have

$$V(x_t) = \sum_{j=0}^{\infty} \beta^j r(x^*_{t+j}, u^*_{t+j}) = r(x^*_t, u^*_t) + \beta \sum_{j=1}^{\infty} \beta^{j-1} r(x^*_{t+j}, u^*_{t+j}) \tag{58}$$

we need to prove

$$\sum_{j=1}^{\infty} \beta^{j-1} r(x^*_{t+j}, u^*_{t+j}) = V(x^*_{t+1}) \tag{59}$$

suppose not given $x^*_{t+1} = \tilde{x}_{t+1}$, we have $\{\tilde{x}_{t+j+1}, \tilde{u}_{t+j}\}_{j=0}^{\infty}$, such that

$$\sum_{j=1}^{\infty} \beta^{j-1} r(x^*_{t+j}, u^*_{t+j}) < \sum_{j=1}^{\infty} \beta^{j-1} r(\tilde{x}_{t+j}, \tilde{u}_{t+j}) \tag{60}$$

then we have

$$V(x_t) < r(x^*_t, u^*_t) + \beta \sum_{j=1}^{\infty} \beta^{j-1} r(\tilde{x}_{t+j}, \tilde{u}_{t+j}) \tag{61}$$

Adding uncertainty and general equilibrium analysis

Now we consider there are many households. Each of them own $s_t(i)$ unit of stock in period t. The stock generated dividend in period t and period t+1 are denoted by d and x. x is draw from a distribution F.

- Household with s unit of stock maximize

$$\log c_t(i) + \beta \int \log(c_{t+1}(i)) f(x) dx$$

with the constraint

$$c_t(i) + Q s_{t+1}(i) = (Q + d) s_t(i) \tag{62}$$

$$c_{t+1}(i) = x s_{t+1}(i) \tag{63}$$

Equilibrium

- Equilibrium is a collect of allocation $\{c_t(i), s_{t+1}(i)\}$ and price Q, given Q and $s_t(i)$, $\{c_t(i), s_{t+1}(i)\}$ solves the household utility maximization problem.
- Market clear

$$\sum s_{t+1}(i) = \sum s_t(i) = \bar{s} \tag{64}$$

$$\sum c_t(i) = \bar{s}d \tag{65}$$

$$\sum c_{t+1}(i) = \bar{s}x \tag{66}$$

Equilibrium

- Solving the household problem

$$\max \log\left[(Q+d)s_t(i) - Qs_{t+1}(i)\right] + \beta\left[\log s_{t+1}(i) + \int \log x dF(x)\right]$$

- f.o.c

$$\frac{Q}{(Q+d)s_t(i) - Qs_{t+1}(i)} = \frac{\beta}{s_{t+1}(i)} \tag{67}$$

or

$$s_{t+1}(i)Q = \frac{\beta}{1+\beta}\left[(Q+d)s_t(i)\right] \tag{68}$$

- we can solve the stock price as

$$Q = \frac{\beta}{1+\beta}(Q+d) \tag{69}$$