Hayashi meets Kiyotaki and Moore: A Theory of Capital Adjustment Costs*

Pengfei Wang
Hong Kong University of Science & Technology

Yi Wen
Federal Reserve Bank of St. Louis
& Tsinghua University

August 2010

Abstract

Firm-level investment is lumpy and volatile but aggregate investment is much smoother and highly serially correlated. This different patterns of investment behavior has been viewed as indicating convex adjustment costs at the aggregate level but non-convex adjustment costs at the firm level. This paper shows that financial frictions in the form of collateralized borrowing at the firm level (Kiyotaki and Moore, 1998) can give rise to convex adjustment costs at the aggregate level, yet at the same time generate lumpiness in plant-level investment. In particular, our model can (i) derive aggregate capital adjustment cost functions that are identical to those assumed by Hayashi (1982) and (ii) explain the weak empirical relationship between Tobin's Q and plant-level investment. However, despite that aggregate adjustment cost functions can be derived from microfoundations, they are subject to the Lucas critique because parameters in such functions may not be structural and policy-invariant.

*We thank Costas Azariadis and Jiajun Miao for comments. The views expressed are those of the individual authors and do not necessarily reflect official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors. Pengfei Wang acknowledges the financial support from Hong Kong Research Grant Council (project #643908). The usual disclaimer applies. Correspondence: Yi Wen, Research Department, Federal Reserve Bank of St. Louis, P.O Box 442, St. Louis, MO, 63166-0442. Phone: 314-444-8559. Fax: 314-444-8731. Email: yi.wen@stls.frb.org.

1 Introduction

It is well known that firm-level investment behaves quite differently from aggregate investment. In particular, firm-level investment is lumpy whereas aggregate investment is much smoother and highly serially correlated (see, e.g., Caballero, 1999). Such a sharp difference in investment dynamics at the plant and aggregate level has often motivated researchers to adopt inconsistent assumptions in explaining investment dynamics: assuming convex adjustment costs in aggregate models and non-convex adjustment costs in micro models. Econometric studies typically find that convex capital adjustment costs (CAC) are consistent with aggregate investment data, but not with firm-level data (e.g., Bloom, 2009).

However, CAC are a widely adopted assumption in dynamic macroeconomic models and have a long tradition in the history of investment theory.[1] This assumption is often needed because a theoretical model without CAC would imply (i) the elasticity of capital supply is the same in both the short run and the long run, i.e., the equilibrium capital stock can be reached instantaneously because of the possibility of an infinite speed of investment rate; and (ii) the relative price of the investment and consumption goods is a constant independent of the relative outputs of the two goods.

Such implications not only are inconsistent with data but also create theoretical difficulties to determine the optimal rate of investment in partial equilibrium models of the firm, which has motivated the early investment literature to adopt CAC (e.g., Lucas, 1967; Gould, 1968). In addition, theory requires CAC to rationalize investment decisions as a function of firm value and replacement costs of capital (Tobin, 1969; Lucas and Prescott, 1971; Hayashi, 1982).[2]

CAC also play an important role in contemporary dynamic stochastic general equilibrium (DSGE) models because (i) it significantly improves the empirical fit of DSGE models with sticky prices (e.g., Christiano, Eichenbaum, and Evans, 2005; Smets and Wouters, 2003); (ii) it is helpful for open-economy models to explain the saving-investment correlations and the home bias puzzle (e.g., Baxter and Crucini 1993); (iii) it is essential to explain the equity premium puzzle in production economies with capital (e.g., Jermann, 1998; Boldrin,

[1] For the early literature that assumes CAC, see Gould (1968), Lucas (1967, 1969), Uzawa (1969), Lucas and Prescott (1971), among others. For a literature survey on investment theory, see Caballero (1999).

[2] Also see Abel (1979, 1983).

Christiano, and Fisher, 2001); (iv) it rationalizes big welfare costs of the business cycle (e.g., Barlevy 2004); and (v) it is key to support news shocks as a credible driving force of the business cycle (e.g., Beaudry and Portier, 2007; Jaimovich and Rebelo, 2009).

However, despite the popularity and apparent "necessity" of CAC in macro models, few microfoundations have been provided in the literature to rationalize CAC, especially the properties imposed on the functional forms of CAC (Hayashi, 1982). This lack of microfoundations unavoidably invites criticisms, such as

(i) empirical analysis based on firm-level data does not find convex adjustment costs important in explaining firm-level investment behavior;[3]

(ii) firm-level investment is lumpy with very little serial correlation, which is inconsistent with convex adjustment costs that smooth out investment over time;[4]

(iii) CAC implies that Tobin's Q should be a sufficient statistics to explain firm-level investment, but firms' investments are far more sensitive to cash flows than to Tobin's Q.[5]

(iv) capital adjustment costs are not observed or recorded in national income accounts, nor in firms' balance sheets.

The goal of this paper is to provide a reconciliation for the apparent inconsistence between micro and macro behaviors of investment. In particular, we show that financial frictions in the form of collateralized borrowing at the firm level can explain convex adjustment costs at the aggregate level and lumpy investment at the firm level.

The typical CAC models assume the following functional form (Hayashi, 1982):

$$K_{t+1} = (1-\delta) K_t + \psi \left(\frac{I_t}{K_t} \right) K_t, \tag{1}$$

where the function $\psi(\cdot)$ is increasing, concave, and homogeneous of degree zero, K_t denotes the existing capital stock, and I_t denotes total investment expenditure as part of a firm's cash flow (CF): $CF = F(K,N) - WN - PI$, where P is the relative price of investment goods. In a one-good economy, $P = 1$. This type of CAC function $\psi(\cdot)$ implies diminishing returns to investment in capital formation—i.e., part of the investment spending is lost and does not become productive capital. Under this type of adjustment cost function, the average Tobin's Q is the same as the marginal Q, which greatly facilitates empirical studies of investment behaviors (Hayashi, 1982).

[3]See e.g, Cooper and Haltiwanger (2006) and Bloom (2009).

[4]See, e.g., Caballero, Engel and Haltiwanger (1995), Cooper, Haltiwanger and Power (1999), Doms and Dunne (1998), and Power (1994).

[5]See, e.g., Hassett and Hubbard (1997) and Caballero (1999).

This form of adjustment costs in equation (1) is equivalent to an alternative formulation of CAC that is also popular in the investment literature. This alternative formulation maintains the neoclassical law of motion for capital, $K_{t+1} = (1-\delta) K_t + \tilde{I}_t$, but redefines a firms' cash flow as

$$F(K_t, N_t) - W_t N_t - P_t C(\tilde{I}_t / K_t) K_t, \tag{2}$$

where the function $C(\cdot)$ denotes total real costs associated with investment expenditure $\tilde{I}_t$ measured in capital units and satisfies the properties $C'(\cdot) > 0$, and $C''(\cdot) > 0$ (see, e.g., Abel, 1982, 1983).

These two forms of adjustment costs formulated in equations (1) and (2) are equivalent, since by redefining $I_t/K_t = C(\tilde{I}_t/K_t)$, we have $\tilde{I}_t/K_t = C^{-1}(I_t/K_t) = \psi(I_t/K_t)$. There are other formulations of capital adjustment costs, but this paper focus on the more standard form defined in equation (1).

Why does aggregate capital accumulation exhibit convex adjustment costs? At least three plausible explanations are available in the literature: (i) Installing new capital takes time and involves sunk costs, delivery lags, and learning (e.g., Cooper and Haltiwanger, 2006). (ii) Capital is firm specific, which makes investment irreversible or partially irreversible (i.e., with resale costs). Irreversibility imposes costs in adjusting the capital stock downward. (iii) Firms are borrowing constrained; hence, they are not able to increase capital at an infinite speed. Borrowing constraints impose costs on adjusting capital upward.

However, two questions naturally arise: Suppose these frictions are explicitly modeled in firms' optimization decisions, (i) would they necessarily give rise to the form of CAC in equation (1)? (ii) If so, do they have the same policy implications as those implied by equation (1)?

These questions are answered in this paper. We show that

(i) If firms' investment projects are subject to idiosyncratic risk (that affects the project's rate of returns) and firms face borrowing constraints with borrowing limit proportional to firm's collateral (capital stock), then the aggregate economy exhibits CAC that is identical in functional form to equation (1).

(ii) Irreversible investment—an important assumption in the investment literature to rationalize convex adjustment costs[6]—is unnecessary for deriving the aggregate CAC function but imposes more structures on the aggregate CAC function. In particular, if investment is

[6] See, e.g., Abel and Eberly (1994, 1996), Pindyck (1991), Dixit (1992), and Dixit and Pindyck (1994).

completely irreversible and the distribution of investment-specific shocks follow the Pareto distribution, then the implied aggregate CAC function becomes the popular Cobb-Douglas form:

$$K_{t+1} = (1-\delta) K_t + b I_t^{\theta} K_t^{1-\theta}, \tag{3}$$

where $\theta \in (0,1)$ is a parameter that depends on the borrowing constraints and distribution of firm-specific shocks.

(iii) Although the aggregate CAC function in equation (1) can be rationalized by the microfoundations provided in this paper, there are still potential dangers in assuming aggregate CAC without explicitly spelling out the microfoundations. For example, when the borrowing limit depends endogenously on the market value of a firm, the implications of government tax policies can be quite different between the aggregate CAC model and the model with microfoundations. That is, traditional CAC are subject to the Lucas' critique because parameters of the CAC function $\psi(\cdot)$ are not structural and policy-invariant.

Our micro-founded model is consistent with the following empirical facts: (i) firm-level investment is lumpy;[7] (ii) firm-level investment has little serial correlation and is insensitive to Tobin's Q; (iii) there do not exists capital adjustment costs in firms' balance sheet; (iv) the elasticity of aggregate capital supply is smaller in the short run and larger in the long run; and (v) the relative price of the investment and consumption goods is not a constant independent of the relative outputs of the two goods.

This paper is related to the work of Carlestrom and Fuerst (1997). Carlestrom and Fuerst (1997) show that the particular type of borrowing constraints studied by Bernanke and Gertler (1989) can imply aggregate capital adjustment costs. The specific financial frictions studied by Bernanke and Gertler (1989) are private information for investment returns and agency costs associated with costly state verification. However, this type of borrowing constraints do not imply an CAC function exactly identical to that in equation (1) because the implied CAC function under agency costs is not homogeneous of degree zero and does not have the desired property that the marginal Q equals the average Q. Hence, our paper differs from this literature in at least three important aspects: (i) The financial friction we consider is based on costly contract enforcement and collateralized borrowing as in Kiyotaki and Moore (1997). More specifically, in the models of Bernanke and Gertler (1989) and Carlestrom and Fuerst (1997), firms do not own capital; instead, they rent capital

[7] The theoretical literature on lumpy investment typically assumes fixed investment costs. Important examples include Veracierto (2002), Thomas (2002), Khan and Thomas (2003, 2008), Gourio and Kashyap (2007), Bachmann et al. (2008), among others.

from entrepreneur households who transform consumption goods into capital by borrowing from other unproductive households. In contrast, firms in our model own capital and they finance investment projects by both internal cash flows and external funds with borrowing limits depending on the firm's collateral value. Thus, we can characterize the relationship between the marginal Q and average Q of a firm, following closely the tradition of Tobin (1969) and Hayashi (1982). (ii) In an agency-cost model, investment is not lumpy because the entrepreneurs always undertake invest in equilibrium. This feature is inconsistent with the data. In contrast, we attempt to quantitatively match the lumpiness of firm-level investment and the correlation between investment rate and Tobin's Q. (iii) We also discuss the policy implications of CAC and show that standard CAC functions are subject to the Lucas critique.

Our work is also related to Lorenzoni and Walentin (2007) and the associated literature that uses simulated data from theoretical models with financial frictions to investigate the quantitative relationship between Tobin's Q and investment (e.g., Gomes, 2001; and others). Lorenzoni and Walentin (2007) show that financial constraints can substantially weaken the correlation between Q and investment, relative to a frictionless benchmark (e.g., Hayashi, 1982). While our model can also explain the weak relationship between Q and investment, our approach differs from theirs in one important aspect: These authors assume CAC in firm's investment technologies, whereas we do not need this assumption. Our paper also differs from theirs in the main focus of the analysis: We try to rationalize and derive CAC from microfoundations.

The rest of the paper is organized as follows. Section 2 presents a benchmark model with a simple form of borrowing constraints and shows how to derive equation (1) from the model. Section 3 studies a model with endogenous borrowing limit and its policy implications. Section 4 conducts quantitative simulations of our microfounded model and examines the model's predictions for the lumpiness of firm-level investment and its correlation with Tobin's Q. Section 5 concludes the paper.

2 The Benchmark Model

2.1 Firms

There are a continuum of competitive firms indexed by $i \in [0, 1]$. Firm i's objective is to maximize its discounted dividends,

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} D_t(i), \tag{4}$$

where D_t represents firm i's dividend in period t and Λ_t the marginal utility of a representative household. The production function has constant returns to scale and is given by

$$Y_t(i) = F(K_t(i), A_t N_t(i)), \tag{5}$$

where A_t represents aggregate labor-augmenting technology that can be either deterministic or stochastic, and $N_t(i)$ and $K_t(i)$ are firm-level employment and capital, respectively. Each firm accumulates capital according to the law of motion,

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i) I_t(i), \tag{6}$$

where $I_t(i)$ denotes investment expenditure and $\varepsilon_t(i) \in R^+$ is an idiosyncratic shock to the marginal efficiency of investment, which has the probability density function $\phi(\varepsilon)$ and cumulative density function $\Phi(\varepsilon)$. For simplicity, assume that this shock is orthogonal to any aggregate shocks. A firm's dividend in period t is hence given by $D_t(i) = Y_t(i) - P_t I_t(i) - W_t N_t(i)$, where $P_t = 1$ denotes the relative price of investment goods and W_t the competitive real wage.

Denote $n_t(i) \equiv N_t(i)/K_t(i)$ as the labor-capital ratio and $f(\cdot) \equiv F(1, \cdot)$ as the output-capital ratio. Given the real wage, the firm's optimal labor demand is determined by the equation, $f_n(A_t n_t(i)) A_t = W_t$. Note that the labor demand function implies that all firms choose the same labor-capital ratio, namely, $n_t(i) = n(w_t, A_t)$ for all i. Firm i's operating profits can then be expressed as $R_t K_t(i) = \max_{N_t(i)} \{Y_t(i) - W_t N_t(i)\}$, where

$$R_t \equiv f(A_t n_t) - w_t n_t \tag{7}$$

is independent of i and the capital stock. Hence, firm's operating profit is proportional to its capital stock. The dividend is then given by $D_t(i) = R_t K_t(i) - I_t(i)$.

We make the following additional assumptions:

(i) Firms' investment is financed by credit and is subject to the borrowing constraint:

$$I_t(i) \le \theta K_t(i), \tag{8}$$

where $\theta > 0$ is a constant. This borrowing constraint specifies that total investment cannot exceed an amount proportional to the existing capital stock. This is similar to the collateralized borrowing assumed by Kiyotaki and Moore (1997). We will make θ endogenous in the next section.

(ii) Firm-level investment may be partially irreversible:

$$K_{t+1}(i) \geq \rho (1-\delta) K_t(i), \tag{9}$$

where the parameter $\rho \in [0,1]$ indicates the degree of irreversibility. For example, if $\rho = 1$, then investment is completely irreversible and equation (9) becomes $I_t(i) \geq 0$. On the other extreme, if $\rho = 0$, then investment is completely reversible and equation (9) becomes $K_{t+1}(i) \geq 0$. Hence, the restriction in equation (9) encompasses both reversible and irreversible investment as special cases. Equation (9) can also be rewritten as

$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)} K_t(i), \tag{10}$$

where $\tilde{\rho} \equiv (1-\rho)(1-\delta)$. Since our general results hold for $\rho = 0$, irreversible investment is not essential for our analysis.

With the definition in equation (7), a firm's maximization problem can be rewritten as

$$\max_{\{i_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left(R_t K_t(i) - I_t(i) \right) \tag{11}$$

subject to equations (6), (8), and (10).

Denote $\{\lambda_t(i), \mu_t(i), \pi_t(i)\}$ as the Lagrangian multipliers of constraints (6), (8), and (10), respectively, the firm's first order conditions for $\{I_t(i), K_{t+1}(i)\}$ are given, respectively, by

$$1 = \varepsilon_t(i)\lambda_t(i) + \pi_t(i) - \mu_t(i), \tag{12}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)\lambda_{t+1}(i) + \theta\mu_{t+1}(i) + \tilde{\rho}\frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} \right\}, \tag{13}$$

plus the complementarity slackness conditions, $\pi_t(i)\left[I_t(i) + \rho\frac{K_t(i)}{\varepsilon_t(i)}\right] = 0$ and $\mu_t(i)[\theta_t K_t(i) - I_t(i)] = 0$. As shown by Wang and Wen (2009) in a similar model, in the case that $\varepsilon_t(i)$ is i.i.d, the Lagrangian multipliers $\{\lambda_t(i),\ \pi_t(i),\ \mu_t(i)\}$ depend only on aggregate states $\mathbf{S}_t$ and $\varepsilon_t(i)$, which implies that the expected value of the Lagrangian multipliers are independent of i; namely, $E_t\lambda_{t+1}(i) = \bar{\lambda}_{t+1}$, $E_t\mu_{t+1}(i) = \bar{\mu}_{t+1}$, and $E_t\pi_{t+1}(i) = \bar{\pi}_{t+1}$. So equation (13) can be rewritten as

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)\bar{\lambda}_{t+1} + \theta\bar{\mu}_{t+1} + \tilde{\rho} \int \frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} d\Phi(\varepsilon) \right\}, \tag{14}$$

which shows that $\lambda_t(i) = \lambda(\mathbf{S}_t) \equiv \lambda_t$ is also independent of i. Since the marginal cost of investment is 1 and the marginal value of newly installed capital stock is λ_t, the market-based measure of Tobin's Q is given by $Q_t = \lambda_t$, which is independent of i.

2.2 Investment Decision Rules

We use a guess-and-verify strategy to derive closed-form decision rules at the firm level. The decision rules are characterized by a cutoff strategy where the cutoff (ε_t^*) pertains to the realization of investment-specific shocks and is defined by the opportunity cost of installing one unit of capital:

$$\lambda_t = \frac{1}{\varepsilon_t^*}. \tag{15}$$

Consider the following possible cases:

Case A: $\varepsilon_t(i) > \varepsilon_t^*$. In this case the marginal efficiency of investment is high. Since the return to investment is high, firms opt to undertake investment up to the borrowing limit, $I_t(i) = \theta_t K_t(i)$. So the constraint (10) does not bind. Hence, we have $\pi_t(i) = 0$. By equation (12), $\mu_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*} + \pi_t(i) - 1 = \frac{\varepsilon_t(i)}{\varepsilon_t^*} - 1 > 0$.

Case B: $\varepsilon_t(i) < \varepsilon_t^*$. In this case the marginal efficiency of investment is low. Since the return to investment is low, firms opt to make minimum amount of investment. So we have $I_t(i) = -\tilde{\rho}\frac{K_t(i)}{\varepsilon_t(i)}$. So the constraint (8) does not bind and we have $\mu_t(i) = 0$. By equation (12), $\pi_t(i) = 1 + \mu_t(i) - \frac{\varepsilon_t(i)}{\varepsilon_t^*} = 1 - \frac{\varepsilon_t(i)}{\varepsilon_t^*} > 0$.

Case C: $\varepsilon_t(i) = \varepsilon_t^*$. By equation (12), $\mu_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*} + \pi_t(i) - 1 = \pi_t(i)$. Suppose $\{\mu_t(i), \pi_t(i)\} > 0$, by the slackness condition we have $I_t(i) = -\tilde{\rho}\frac{K_t(i)}{\varepsilon_t(i)}$ and $I_t(i) = \theta K_t(i)$, which leads to a contradiction. Hence, we must have $\mu_t(i) = \pi_t(i) = 0$. In this marginal case, equation (12) implies $\lambda_t = \frac{1}{\varepsilon_t^*}$, which confirms that the cutoff is indeed given by equation (15). Without loss of generality, we assume that in this marginal case a firm undertakes maximum invest.

Notice that from an individual firm's own perspective, Tobin's Q is measured by $q_t(i) \equiv \frac{\varepsilon_t(i)}{\varepsilon_t^*}$. A firm will undertake positive investment if $q(i) \geq 1$, otherwise the firm disinvest or remains inactive. However, because markets are incomplete and the idiosyncratic shocks are not observable (or insured) through markets, the market-based measure of Tobin's Q is $\frac{1}{\varepsilon_t^*}$, which is independent of $\varepsilon_t(i)$.

Based on the above analysis, the Lagrangian multipliers satisfy $\mu_t(i) = \max\{q_t(i) - 1, 0\}$ and $\pi_t(i) = \max\{1 - q_t(i), 0\}$. The firm's decision rules for investment and capital accumulation are given by

$$I_t(i) = \begin{cases} \theta K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ -\tilde{\rho}\frac{K_t(i)}{\varepsilon_t(i)} & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \tag{16}$$

$$\frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{R_{t+1} + \frac{(1-\delta)}{\varepsilon_{t+1}^*} + O(\varepsilon_{t+1}^*)\right\}, \tag{17}$$

where the implicit function $O(\cdot)$ in the last equation is defined by

$$\begin{aligned} O(\varepsilon_{t+1}^*) &\equiv E_t\left[\theta\mu_{t+1}(i) + \tilde{\rho}\frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)}\right] \qquad (18) \\ &= \theta\int_{\varepsilon_{t+1}(i)\geq\varepsilon_{t+1}^*} \frac{\varepsilon_{t+1}(i) - \varepsilon_{t+1}^*}{\varepsilon_{t+1}^*} d\Phi(\varepsilon) + \tilde{\rho}\int_{\varepsilon_{t+1}(i)<\varepsilon_{t+1}^*}\left(\frac{1}{\varepsilon_{t+1}(i)} - \frac{1}{\varepsilon_{t+1}^*}\right) d\Phi(\varepsilon). \end{aligned}$$

The investment function (16) indicates that firm-level investment is lumpy and lacks serial correlations. Each firm in any period has only probability $1 - \Phi(\varepsilon_t^*)$ to undertake positive investment and probability $\Phi(\varepsilon_t^*)$ to remain inactive (or disinvest). These probabilities are determined by aggregate economic conditions that influence the cutoff (ε_t^*) and independent of each firm's investment history (which is highly idiosyncratic). Also, such lumpiness is independent of the value of $\tilde{\rho}$; namely, the lumpiness does not hinge on irreversibility.

Notice that $O(\cdot)$ is the option value of one unit of installed capital:[8] If the firm receives a favorable shock in the next period, one unit of installed capital can expand firm's borrowing capacity by θ units and each additional unit can bring a net profit of $q(i) - 1$ $(= \frac{\varepsilon(i)-\varepsilon^*}{\varepsilon^*})$ units. This case happens with probability $\int_{\varepsilon\geq\varepsilon_{t+1}^*} d\Phi(\varepsilon)$. In the case of an unfavorable shock,

[8] Note that we have used the orthogonality condition between idiosyncratic shocks and aggregate uncertainty to derive equation (17).

the firm can disinvest by $\tilde{\rho} \geq 0$ units and each unit of saving can be transform into $\frac{1}{\varepsilon}$ units of consumption goods. By doing so, the firm can increase net profit by $\frac{1-q}{\varepsilon}$ $\left(= \frac{1}{\varepsilon(i)} - \frac{1}{\varepsilon^*}\right)$ units.

Hence, equation (17) implies that the optimal level of investment is determined to the point where the marginal cost $\left(\frac{1}{\varepsilon_t^*}\right)$ equals the marginal benefits (= the marginal product of capital + the value of non-depreciated capital + the option value of capital). Because the optimal level of investment depends on the expected returns, which in turn depend on the probability weights of the different cases considered above (i.e., the cutoff ε_t^*), equation (17) says that a firm chooses the optimal cutoff ε_t^* (as an implicit function of aggregate economic conditions) so that the marginal cost of investment equals the expected marginal gains.

Equation (17) also shows that the optimal cutoff ε_t^* is independent of i, namely, it is the same across all firms. This is the consequence of the assumption that $\varepsilon(i)$ is i.i.d. More specifically, the optimal cutoff is independent of firms' investment rate and existing capital stock. This property enables us to characterize aggregate investment dynamics in a tractable manner without the need of using numerical methods (such as in Krusell and Smith, 1998).

2.3 Properties of Aggregate Investment Function

Integrating the firm-level decision rules by the law of large numbers, the aggregate investment, aggregate capital stock, and the optimal cutoff are determined jointly by the following three equations:

$$\frac{I_t}{K_t} = \theta\left[1 - \Phi(\varepsilon_t^*)\right] - \tilde{\rho} \int_{\varepsilon < \varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon); \tag{19}$$

$$K_{t+1} = (1-\delta)K_t + \theta K_t \int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) - \tilde{\rho} K_t \int_{\varepsilon < \varepsilon_t^*} d\Phi(\varepsilon), \tag{20}$$

and equation (17); where equation (19) is derived from equation (16) and equation (20) from equation (6). It can be confirmed by the eigenvalue method that this three-equation dynamic system has a unique saddle-path steady state. Hence, given the stochastic process of $\{R_t, \Lambda_t\}$, the equilibrium path of $\{I_t, K_{t+1}, \varepsilon_t^*\}$ is uniquely determined.[9]

Equation (19) suggests that the aggregate investment rate is fully determined by ε_t^*. Given the parameters θ and $\tilde{\rho}$, this equation also define the cutoff as an implicit function of

[9] See Wang and Wen (2009) for a general-equilibrium analysis of a similar model with a household sector.

the investment rate: $\varepsilon_t^* = \varepsilon^*(\frac{I_t}{K_t})$. Therefore, equation (20) can be written as

$$K_{t+1} = (1-\delta)K_t + \varphi(\frac{I_t}{K_t})K_t, \tag{21}$$

where

$$\varphi(\frac{I_t}{K_t}) \equiv \theta \int_{\varepsilon \geq \varepsilon^*(\frac{I_t}{K_t})} \varepsilon d\Phi(\varepsilon) - \tilde{\rho} \int_{\varepsilon < \varepsilon^*(\frac{I_t}{K_t})} d\Phi(\varepsilon) \tag{22}$$

is an implicit function of aggregate investment rate.

Proposition 1 *The implicit function $\varphi(\cdot)$ is increasing, strictly concave, and homogenous of degree zero in $\{I_t, K_t\}$.*

Proof. See Appendix I. ■

2.4 Equivalence

If we define the market value of one unit of newly installed capital (or Tobin's Q) of a firm as

$$Q_t \equiv \lambda_t = \frac{1}{\varepsilon_t^*}, \tag{23}$$

using equation (19), we can simplify the implicit function $O(\varepsilon_t^*)$ in equation (18) as

$$\begin{aligned} O(\varepsilon_t^*) &= \frac{1}{\varepsilon_t^*}\left[\theta \int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) - \tilde{\rho} \int_{\varepsilon < \varepsilon_t^*} d\Phi(\varepsilon)\right] - \left\{\theta[1-\Phi(\varepsilon_t^*)] - \tilde{\rho}\int_{\varepsilon < \varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon)\right\} \\ &= Q_t \varphi(i_t) - i_t. \end{aligned} \tag{24}$$

Therefore, using the defined functions for $\{\varphi(\cdot), \varphi'(\cdot), O(\cdot), Q\}$, the system of equations that solve for aggregate investment rate (i_t), the capital stock (K_{t+1}), and Q_t are given by

$$Q_t \varphi'(i_t) = 1, \tag{25}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{R_{t+1} + Q_{t+1}(1-\delta) + Q_{t+1}\varphi(i_{t+1}) - i_{t+1}\right\}, \tag{26}$$

$$K_{t+1} = (1-\delta)K_t + \varphi(i_t)K_t. \tag{27}$$

Now consider a standard representative-agent macro model of CAC (e.g., Hayashi, 1982), where a representative-firm solves (taking as given the marginal product of capital R_t)

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} (R_t K_t - I_t) \tag{28}$$

subject to

$$K_{t+1} = (1-\delta) K_t + \psi(\frac{I_t}{K_t}) K_t. \tag{29}$$

Define Q_t (Tobin's Q) as the Lagrangian multiplier for the constraint (29) and $i_t \equiv \frac{I_t}{K_t}$ as the investment rate, the first-order conditions for $\{I_t, K_{t+1}\}$ are given, respectively, by

$$Q_t \psi'(i_t) = 1 \tag{30}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{R_{t+1} + (1-\delta) Q_{t+1} + Q_{t+1} [\psi(i_{t+1}) - \psi'(i_{t+1}) i_{t+1}]\}. \tag{31}$$

Using equation (30), we can rewrite equation (31) as

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{R_{t+1} + (1-\delta) Q_{t+1} + Q_{t+1} \psi(i_{t+1}) - i_{t+1}\}. \tag{32}$$

Notice that the system of equations (29), (30), and (32) are identical to the system of equations (25)-(27) if the two CAC functions, $\varphi(\cdot)$ and $\psi(\cdot)$, are identical. Since the equivalence between our micro-founded model and the aggregate CAC model holds regardless of the value of $\tilde{\rho}$ in the micro-founded model, the equivalence result is established without relying on the assumption of irreversible investment. The key assumption for the equivalence is the collateralized borrowing constraint (8).

The equivalence result holds regardless of the exogenous driving processes of $\{R_t, W_t, \Lambda_t\}$. That is, the two models are identical not only in the steady state but also along any transitional dynamic path. For example, the impulse responses of the two models are completely identical under either aggregate technology shocks that affect $\{R_t, W_t\}$ or aggregate demand shocks that affect Λ_t.

Example To further illustrate the equivalence result, suppose the distribution of ε is Pareto with support $[1, \infty)$ and shape parameter $\eta > 1$, namely, $\Phi(\varepsilon) = 1 - \varepsilon^{-\eta}$; and assume that investment is completely irreversible, $\tilde{\rho} = 0$. With these assumptions, equations

(19) and (20) become, respectively, $\frac{I_t}{K_t} = \theta \varepsilon_t^{*-\eta} = \theta Q_t^{\eta}$ and $K_{t+1} = (1-\delta)K_t + \frac{\eta\theta}{\eta-1}\varepsilon_t^{*1-\eta}K_t$. The adjustment cost function is $\varphi(\frac{I_t}{K_t}) = \frac{\eta\theta^{\frac{1}{\eta}}}{\eta-1}\left(\frac{I_t}{K_t}\right)^{\frac{\eta-1}{\eta}}$, which is homogeneous of degree zero and satisfies $\varphi'(\frac{I_t}{K_t})Q_t = 1$. Substituting out ε_t^*, equation (32) then becomes $Q_t = E_t\beta_{t+1}\left\{R_{t+1} + Q_{t+1}(1-\delta) + \frac{1}{\eta-1}\frac{I_{t+1}}{K_{t+1}}\right\}$; and the law of capital accumulation becomes

$$K_{t+1} = (1-\delta)K_t + \frac{\eta\theta^{\frac{1}{\eta}}}{\eta-1} I_t^{\frac{\eta-1}{\eta}} K_t^{\frac{1}{\eta}}, \tag{33}$$

which is identical to equation (3) and has the familiar Cobb-Douglas form commonly assumed in the macro literature.

2.5 Intuition

In representative-agent CAC models, capital adjustment costs imply that aggregate investment rate is sluggish in responding to macroeconomic environmental changes because of diminishing returns to investment in capital formation. In other words, because $\psi(\cdot)$ is concave, aggregate investment responds to a higher future capital productivity (R_{t+1}) less elastically than it would be otherwise. As a result, the optimal capital stock can only be reached through multiple periods of investment at a finite speed instead of through a single-period investment at a infinite speed.

In our heterogeneous-agent model, firm-level investment is lumpy because a firm undertakes either a large amount of positive investment (called "active" firms) or a large amount of negative investment (called "inactive" firms), depending on the idiosyncratic shock to the rate of return to investment in a particular period. However, despite the lumpiness of firm-level investment, aggregate investment is sluggish. Aggregate investment in our model has two margins, an intensive margin that depends on each firm's maximum investment level (θ) and an extensive margin that depends on the number of active firms (ε_t^*) in a period. Equation (19) shows that the aggregate investment rate depends on θ (the intensive margin) and the proportion of active firms, $1 - \Phi(\varepsilon^*) = \Pr[\varepsilon \geq \varepsilon^*]$ (the extensive margin, assuming $\tilde{\rho} = 0$ for a moment). However, the extensive margin is determined by the optimal cutoff ε_t^*, which behaves sluggishly because by equation (17) the inverse of the cutoff ε_t^*—Tobin's Q—is a slow moving (weighted) average of expected future marginal products ($R_{t+j}, j = 1, 2, ...$) as well as the option values of capital ($O_{t+j}, j = 1, 2, ...$). Hence, when R_{t+1} changes, the optimal level of aggregate capital stock cannot be reached through a single-period aggregate investment because the extensive margin (Tobin's Q) adjusts slowly over time (since an increase in R_{t+1} has only a small impact on the cutoff).

Aggregate investment in the representative CAC model depends fully and positively on Tobin's Q because Q contains all information about the marginal costs and benefits of investment—a higher value of capital is required for a higher investment rate when the marginal cost of investment is increasing. However, it is well known that this Q-theory of investment has not been fared well empirically. Variables such as firms' cash flows are always found more important in explaining firm-level investment than the average Q (see, e.g., Hassett and Hubbard, 1997).

Our approach provides an explanation for this apparent failure of the Q-theory. In our model, firm-level investment is financed by cash flows but limited by external borrowing. Cash flows captures the idiosyncratic shocks $\varepsilon(i)$ while the market based measure of Q does not, which makes firms' cash positions and net worth more important than the market value of Tobin's Q in determining the level of investment. On the other hand, without borrowing constraints, only the most productive firm (or most efficient firm with the highest draw of $\varepsilon(i)$) will undertake investment in each period; the model then degenerates to a representative-firm model in which Q is a sufficient statistics for determining firm's investment. Hence, both idiosyncratic shocks and borrowing constraints are important in rendering firm-level investment insensitive to Q.

Yet at the aggregate level, total investment depends positively and fully on Q for the following reasons: Since a firm's investment is constrained by the firm's capital stock, only a fraction of firms (i.e., the most efficient firms) will undertake positive investment and the rest of firms remain inactive in each period. Thus, an increase in the aggregate stock of capital requires a greater proportion of active firms. This is possible in equilibrium only if the market value of capital (Q) increases (or the cutoff ε^* decreases) so that more firms (including the less efficient ones) also find investment profitable. In other words, the less efficient firms raise the aggregate marginal cost of investment, hence calling for a higher Q to balance it in equilibrium. Therefore, aggregate investment has a close relationship to Q. This explains why empirical work based on micro data will tend to find firms' cash flows more important than Q in determining the rate of firm investment in the short run, but aggregate data and long-run analysis will tend to find Q important and significant in determining aggregate investment (see, e.g., Caballero, 1999; Cooper and Haltiwanger, 2006).

3 Endogenous Borrowing Constraints

In the benchmark model, the borrowing limit is assumed to be a fixed proportion of the existing capital stock. In general, firms' borrowing limits may depend on the value of the collateral (Kiyotaki and Moore, 1998). That is, the parameter θ may be endogenous. To relax this assumption, consider the following borrowing constraint with endogenous credit limits:

$$I_t(i) \leq \theta Q_t(i) K_t(i), \tag{34}$$

where $Q_t(i)$ denotes the market value of firm i's existing capital stock and $\theta > 0$ is a parameter.[10]

Proposition 2 *Assuming $\tilde{\rho} = 0$ for simplicity (without loss of generality), the optimal investment and capital accumulation policies of a firm are given by*

$$I_t(i) = \begin{cases} \theta Q_t K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \tag{35}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta) Q_{t+1} + \theta Q_{t+1} \int_{\varepsilon_{t+1}(i) \geq \varepsilon_{t+1}^*} \left[\frac{\varepsilon_{t+1}(i)}{\varepsilon_{t+1}^*} - 1 \right] d\Phi(\varepsilon) \right\}, \tag{36}$$

where $Q_t = \frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} v_{t+1}(i)$. The value function of a firm per unit of capital ($v_t(i)$) is given by

$$v_t(i) = \begin{cases} R_t + (1-\delta) Q_t + \theta Q_t [Q_t \varepsilon_t(i) - 1] & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ R_t + (1-\delta) Q_t & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}. \tag{37}$$

Proof. See Appendix II. ■

Hence, the decision rules take exactly the same form as those in the benchmark model (except here the parameter θ in the benchmark model is replaced by θQ_t). Note that the market value of firm i's existing capital stock and the marginal value of one unit of newly installed capital are the same across firms (i.e., independent of i), as in the benchmark model. That is, $Q_t(i) = Q_t$ and all firms have the same Tobin's Q from the perspective of the markets.[11]

[10] For example, if the non-depreciated capital stock is fully collateralized, then $\theta = 1 - \delta$.

[11] As mentioned earlier in the previous section, because markets are incomplete in the model, idiosyncratic shocks to a firm's investment return are uninsured. Hence, $q_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*}$ cannot be used by the market to determine a firm's Q. This is why the market-based measure of Q is $\frac{1}{\varepsilon_t^*}$, instead of $\frac{\varepsilon(i)}{\varepsilon^*}$.

By the law of large numbers, the aggregate investment is given by

$$I_t = \theta K_t Q_t[1 - \Phi(\varepsilon_t^*)]. \tag{38}$$

Since $Q_t = \frac{1}{\varepsilon_t^*}$, the above equation defines the implicit function $Q_t = Q\left(I_t/K_t\right)$. We can use this implicit equation to rewrite equation (36) as

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{R_{t+1} + (1-\delta)Q_{t+1} + \theta Q_{t+1}^2 \int_{\varepsilon \geq \frac{1}{Q(I_{t+1}/K_{t+1})}} \varepsilon d\Phi(\varepsilon) - \frac{I_{t+1}}{K_{t+1}}\right\} \tag{39}$$

Similarly, the aggregate law of motion for capital accumulation is given by

$$K_{t+1} = (1-\delta)K_t + \Psi(\frac{I_t}{K_t})K_t, \tag{40}$$

where

$$\Psi(\frac{I_t}{K_t}) \equiv \theta Q(I_t/K_t) \int_{\varepsilon \geq \frac{1}{Q(I_t/K_t)}} \varepsilon d\Phi(\varepsilon). \tag{41}$$

Proposition 3 *For any probability density function $\phi(\varepsilon)$ that satisfies $\phi'(\varepsilon) \leq 0$, the implicit function $\Psi(\cdot)$ in equation (40) is increasing, concave, and homogenous of degree zero in $\{I_t, K_t\}$.*

Proof. See Appendix III. ■

Example Many standard distributions, such as Pareto, Exponential, and Uniform distribution, satisfy the property $\phi'(\varepsilon) \leq 0$. As an example, consider Pareto distribution, $\Phi(\varepsilon) = 1 - \varepsilon^{-\eta}$ with $\eta > 1$. Equation (38) becomes $\frac{I_t}{K_t} = \theta \varepsilon_t^{*-\eta-1}$, and the capital accumulation equation becomes $K_{t+1} = (1-\delta)K_t + \theta \frac{\eta}{\eta-1}\varepsilon_t^{*-\eta}$. Combining these two equations together imply

$$K_{t+1} = (1-\delta)K_t + \varphi_0 K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}} \tag{42}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left[R_{t+1} + (1-\delta)Q_{t+1} + \frac{1}{\eta-1}\frac{I_{t+1}}{K_{t+1}}\right] \tag{43}$$

$$\frac{I_t}{K_t} = \theta Q_t^{\eta+1}, \tag{44}$$

where $\varphi_0 \equiv \frac{\eta}{\eta-1}\theta^{\frac{1}{\eta+1}} > 0$. Thus, similar to the benchmark model, under irreversible investment, Pareto distribution gives a reduced-form Cobb-Douglas capital adjustment function $\Psi(I_t/K_t) = \varphi_0 \left(\frac{I_t}{K_t}\right)^{\frac{\eta}{\eta+1}}$.

Therefore, we have shown that borrowing constraints at the firm level can fully rationalize the CAC function (1). In other words, the specific form aggregate capital adjustment costs assumed by Hayashi (1982) and others in the existing literature can be derived from microfoundations with financial frictions that hinder firms' ability to borrow. However, there exist subtle but important differences between the exogenous borrowing limit model and the endogenous borrowing limit model, as shown below.

3.1 Non-Equivalence

With an endogenous borrowing limit, the equivalence between the microfounded heterogeneous-firm model and the representative-agent CAC model holds only so long as equation (1) is concerned. Unlike the benchmark model, however, the endogenous borrowing limit model and the CAC model are not fully equivalent because the trajectories of investment and capital stock in the endogenous borrowing limit model are no longer identical to those implied by the CAC model. That is, even though the two models share the same law of motion for aggregate capital accumulation as in equation (29), the first-order conditions in equations (30) and (32) (derived in the representative-firm model) no longer hold in the microfounded model with endogenous borrowing limits.

The source of the discrepancy stems from the endogeneity of the borrowing constraints in equation (34), where the market value of capital, $Q(I_t/K_t)$, is positively affected by the rate of aggregate investment. Hence, the more investment each firm undertakes, the higher is the value of a firm, and thus the more creditworthy each firm becomes. However, this type of credit externality is not internalized by firms because Q is an aggregate market price taken as given by individual firms. As a result, the micro-founded model appears to have an insufficient investment level relative to the counterpart representative-agent CAC model.

The following proposition shows that the credit externality in the endogenous borrowing limit model is equivalent to an aggregate "investment externality" in a conventional CAC model, where the source of the aggregate investment externality is a social rate of return to the average investment that individual firms take as given.

Proposition 4 *The heterogenous-firm model with an endogenous borrowing limit is obser-*

vationally equivalent to the following representative-firm CAC model with investment externalities:

$$K_{t+1} = (1-\delta)K_t + \tilde{\Psi}(\bar{\imath}_t, i_t)K_t, \tag{45}$$

where $\bar{\imath}_t \equiv \frac{\bar{I}_t}{\bar{K}_t}$ *denotes the average investment-to-capital ratio in the economy that the representative firm takes as given, and the CAC function* $\tilde{\Psi}(\cdot,\cdot)$ *is increasing and concave in* $\{\bar{\imath}_t, i_t\}$ *and satisfies the decomposition:* $\tilde{\Psi}(\bar{\imath}_t, i_t) = \theta Q(\bar{\imath}_t)\varphi(i_t)$*, where the function* $\varphi(\cdot)$ *satisfies*

$$\varphi(i_t) = \int_{\varepsilon \geq \varepsilon^*(i_t)} \varepsilon d\Phi(\varepsilon). \tag{46}$$

Proof. *See Appendix IV.* ■

Example As an example, consider the microfounded model with Pareto distribution. The model's equilibrium is characterized by equations (42), (43), and (44). Now consider a representative-firm CAC model with investment externality $\left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a$:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[R_t K_t - I_t\right] \tag{47}$$

subject to

$$K_{t+1} = (1-\delta)K_t + \varphi_0 \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_t^{1-b} I_t^b, \tag{48}$$

where $\varphi_0 = \frac{\eta}{\eta-1}\theta^{\frac{1}{\eta+1}}$, $a = \frac{1}{\eta(\eta+1)}$, $b = \frac{\eta-1}{\eta}$, and $\frac{\bar{I}_t}{\bar{K}_t}$ denotes the average investment rate in the economy that the representative firm takes as given. Denoting Q_t as the Lagrangian multiplier for the constraint. The first order conditions with respect I_t and K_{t+1} are given, respectively, by

$$Q_t \varphi_0 b \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_t^{1-b} I_t^{b-1} = 1 \tag{49}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)Q_{t+1} + Q_{t+1}\varphi_0(1-b)\left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_{t+1}^{-b} I_{t+1}^b \right\}. \tag{50}$$

Imposing the equilibrium condition, $\frac{\bar{I}_t}{\bar{K}_t} = \frac{I_t}{K_t}$, and plugging in the values of $\{\varphi_0, a, b\}$, equation (48) becomes

$$K_{t+1} = (1-\delta)K_t + \frac{\eta\theta^{\frac{1}{\eta+1}}}{\eta-1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}}; \tag{51}$$

equation (49) becomes

$$\frac{I_t}{K_t} = \theta Q_t^{1+\eta}; \tag{52}$$

and equation (50) changes to

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta) Q_{t+1} + \frac{1}{\eta - 1} \frac{I_{t+1}}{K_{t+1}} \right\}. \tag{53}$$

The above three equations are identical to equations (42) though (44) in the microfounded model.

3.2 Policy Implications

The above analysis suggests the perils of relying on the apparent equivalence in the CAC function to rationalize capital adjustment cost models. In particular, the endogenous credit limit model reveals that the assumption of aggregate CAC is subject to the Lucas critique because parameters in the CAC function are not structural and policy-invariant. As an example for the different policy implications between the two models, we have the following

Proposition 5 *The optimal steady-state capital tax rate in the representative-agent CAC model is zero while it is negative in the endogenous credit limit model.*

Proof. See Appendix V. ■

The intuition behind this proposition is simple. The endogenous credit limit model features a positive credit externality on firm's investment. Because firms take the borrowing limit as exogenous while it is endogenously determined by the market equilibrium, the competitive equilibrium features suboptimal investment and leads to insufficient capital stock. Alternatively, since the model is equivalent to an representative-agent CAC model with positive investment externalities, the investment level determined by a representative firm in a competitive equilibrium is lower than optimal. Therefore, a negative capital tax rate to encourage more investment improves social welfare.

4 Tobin's Q and Firm-Level Investment

This section solves for a general-equilibrium version of our microfounded investment model and use simulated data from the model to investigate relationship between firm-level investment and Tobin's Q. Since a firm's investment rate depends on the firm value and other macro economic variables such as the real wage, a general-equilibrium model is required.

A representative consumer (owner of firms) solves

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{\log C_t - a_L N_t\} \tag{54}$$

subject to

$$C_t \leq W_t N_t + \Pi_t, \tag{55}$$

where Π_t denotes lump-sum profit income from all firms. Notice that, for simplicity, the household does not save. If we introduce an equity market where households can buy firms' shares, the results would be identical. Denoting Λ_t as the Lagrange multiplier of the household's budget constraint, the first-order conditions of the representative household are given by

$$\Lambda_t = \frac{1}{C_t}, \tag{56}$$

$$\frac{(1-\alpha)Y_t}{N_t}\frac{1}{C_t} = a_L. \tag{57}$$

The problem of the firm is identical to that in the previous section with endogenous borrowing limits. Namely, firms' decision rules are given by equations (35) through (37), and the following relationships hold: $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$, $W_t = (1-\alpha)\frac{Y_t}{N_t}$, and $R_t = \alpha \frac{Y_t}{K_t}$. Under the assumption of Pareto distribution, the competitive general equilibrium of the aggregate economy is characterized by these three relationships, plus equations (42), (43), (44), (56), and (57). This system of 8 equations determines the equilibrium path of $\{C_t,N_t,Y_t,I_t,K_{t+1},Q_t,W_t,R_t\}$. The equilibrium cutoff is determined by $\varepsilon_t^* = Q_t^{-1}$. It can be easily confirmed by the eigenvalue method that the model has a unique saddle-path steady state near the steady state. We solve the model by log-linearization around the steady state under the assumption that the aggregate productivity (A_t) evolves according to the law of motion,

$$\log A_t = \rho \log A_{t-1} + \sigma \xi_t, \tag{58}$$

where ξ_t is i.i.d. with standard deviation normalized to 1.

Calibration. We calibrate the model at a quarterly frequency by setting the time discounting factor $\beta = 0.99$, the capital's income share $\alpha = 0.3$, the persistence of technology shock $\rho = 0.98$, and the standard deviation of innovation $\sigma = 0.0072$ (as in standard RBC literature). Since a_L does not enter the model's log-linear dynamic system, we choose a_L

such that $N = 1$ in the deterministic steady state. The other three parameters, namely, the depreciation rate of capital δ, the borrowing limit θ, and the Pareto distribution parameter η, are chosen so that the model matches the distribution of firm-level investment. The parameter values are summarized in Table 1.

Table 1. Parameter Values

β	α	a_L	ρ	σ	δ	θ	η
0.99	0.3	1.097	0.98	0.0072	0.032	0.08	2.4

We follow Cooper and Haltiwanger (2006) by defining $i_t(i) = \frac{K_{t+1}(i)-(1-\delta)K_t(i)}{K_t(i)}$ as a firm's investment rate. The annual investment rate in the model is calculated by simulation and time aggregation. We simulate 200,000 quarters of data. We first use a general equilibrium model to obtain the cutoff ε_t^*; we then draw 200,000 independent draw of $\varepsilon_t(i)$ for a typical firm by normalizing its initial capital stock. We then calculate the annual investment rate for $\tau = 1, 2, ...50, 000$ by

$$i_\tau^A = \frac{K_{4\tau} - (1-\delta)^4 K_{4*(\tau-1)}}{K_{4*(\tau-1)}}. \tag{59}$$

(More details of the simulation procedure can be found in Appendix VI). The statistics for the annualized investment rate i_t^A are reported in Table 2, where the empirical counterpart are based on statistics reported by Cooper and Haltiwanger (2006, p. 615, Table 1).

Table 2. Summary Statistics for Annualized Investment Rate

Investment Rate	$i^A \leq 0$	$0 < i^A \leq 20\%$	$i^A > 20\%$	$E[i^A]$	$std[i^A]$	$\frac{E[i^A \mid i^A \geq 0.2]}{E[i^A]}$	$\rho(i_t^A, i_{t-1}^A)$
Model	17.7%	63.6%	18.7%	12.0%	32.0%	56.3%	0.47%
Data	18.5%	62.9%	18.6%	12.2%	33.7%	50.0%	5.8%

The table shows that our microfounded model is able to match the basic features of firm-level investment dynamics reported by Cooper and Haltiwanger (2006). For example, out model predicts that (i) In any given year, about 18% of firms are inactive (making zero or negative investment), about 19% of firms undertake big investment projects (with values exceeding 20 percent of the existing capital stock), and the average investment rate is about 12% a year. These predictions match the data almost exactly. (ii) The standard deviation of investment rate is 32% whereas it is 34% in the data. (iii) Firm-level investment is not serially correlated. The model predicts an autocorrelation of 0.0047 while this value is 0.058

in the data. Therefore, our model does a very good job in explaining the lumpiness and lack of serial correlations in firm's investment behavior.

Another well-known empirical puzzle is that firm-level investment is not sensitive to Tobin's Q but standard CAC models predicts that Tobin's Q is a sufficient statistic to determine firms' investment rate. To test if our model provides a plausible resolution to this puzzle, we run regressions for both firm-level investment rate on Q_t (Tobin's Q) for two models, one for our microfounded model and another for the representative-firm CAC model. The regression has a constant as an independent variable. We report R^2 of the regressions. If Tobin's Q is a sufficient statistic for investment, the value of R^2 should be 1; if not, the value should be far less than 1.

Table 3. Relationship between Investment and Tobin's Q

	CAC Model		Our model	
Variable	Coefficient	Standard Error	Coefficient	Standard Error
Constant	-0.022	2.0e-006	-0.031	0.011
Q_t	0.0596	3.78e-006	0.114	0.020
R^2	1.0000		0.00016	

The table shows that in both models, investment rate depends positively on Q_t. However, in the conventional CAC model Q_t is a sufficient statistic for investment because the $R^2 = 1$; whereas in our model Q_t has very little explanatory power on investment—the R^2 is close to zero, as has been noticed by the large existing empirical literature.

5 Conclusion

The popular assumption of CAC in DSGE models is consistent with aggregate investment behaviors but inconsistent with micro evidence. This paper tempts to provide microfoundations to rationalize CAC through financial frictions. In particular, we show that collateralized borrowing due to contract enforcement problem at the firm level can generate lumpy investment at the firm level and CAC functions at the aggregate level. Therefore, we provide a justification for the large dynamic macro literature that assumes CAC. However, we also point out the potential perils in assuming aggregate CAC in DSGE models: CAC may be subject to the Lucas critique because the parameters in CAC functions are not necessarily policy invariant. Consequently, policy analysis based on CAC models may be invalid and misleading. An example of optimal capital tax is given in this paper to illustrate this point. We believe that the analysis also applies to optimal monetary policies. For example, the new Keynesian sticky-price literature often assumes CAC to better match the aggregate data. In such models, optimal monetary policies often face trade-offs between output gap and inflation. However, if CAC arises from endogenous borrowing constraints and money supply affects banks' lending policies and credit availability, optimal monetary policy may put more weight on output gap relative to inflation than it would otherwise in a CAC model. These issues are worth further studies.

Appendix I. Proof of Proposition 1

Proof. Denoting $i_t \equiv \frac{I_t}{K_t}$ and taking derivative of the function $\varphi(\cdot)$ in equation (22) with respect to i_t gives

$$\varphi'(i_t) = \left[-\theta\varepsilon_t^*\phi(\varepsilon_t^*) - \tilde{\rho}\phi(\varepsilon_t^*)\right]\frac{\partial\varepsilon_t^*}{\partial i_t}, \tag{60}$$

where $\phi(\varepsilon)$ denotes the PDF of ε. Differentiating equation (19) with respect to $i_t \equiv \frac{I_t}{K_t}$, we have

$$\frac{\partial i_t}{\partial\varepsilon_t^*} = -\theta\phi(\varepsilon_t^*) - \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*). \tag{61}$$

The above two equations together imply

$$\varphi'(i_t) = \frac{\theta\varepsilon_t^*\phi(\varepsilon_t^*) + \tilde{\rho}\phi(\varepsilon_t^*)}{\theta\phi(\varepsilon_t^*) + \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*)} = \varepsilon_t^* > 0. \tag{62}$$

Differentiating this equation with respect to i_t again and using equation (61) gives

$$\varphi''(i_t) = \frac{\partial\varepsilon_t^*}{\partial i_t} = \frac{1}{-\theta\phi(\varepsilon_t^*) - \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*)} < 0. \tag{63}$$

Therefore, the function $\varphi(i_t)$ is increasing and strictly concave in i_t. Since $\varphi(i_t)$ depends only on the investment to capital ratio, it is homogenous of degree zero in $\{I_t, K_t\}$. ■

Appendix II. Proof of Proposition 2

Proof. Denote $V\left[K_t(i), \varepsilon_t(i)\right]$ as the value function of firm i with capital stock $K_t(i)$. Based on the analysis of Hayashi (1982), we conjecture that a firm's value is linearly homogeneous in its capital stock because of constant returns to scale production technology:

$$V[K_t(i), \varepsilon_t(i)] = v\left[\varepsilon_t(i)\right]K_t(i) \equiv v_t(i)K_t(i). \tag{64}$$

We will verify later that this conjecture is correct. Define $\bar{v}_t \equiv Ev_t(i) = \displaystyle\int v_t(\varepsilon)d\Phi(\varepsilon)$ as the average value of the firm across states and $i_t(i) \equiv \frac{I_t(i)}{K_t(i)}$ as the firm's investment rate. Firm i solves the following dynamic programing problem,

$$v_t(i)K_t(i) = \max_{K_{t+1}(i), I_t(i)}\left\{R_tK_t(i) - I_t(i) + \beta E_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}v_{t+1}(i)K_{t+1}(i)\right]\right\} \tag{65}$$

subject to

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i)I_t(i) \tag{66}$$

$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)}K_t(i) \tag{67}$$

and the borrowing constraint in equation (34). To simplify the analysis, assume $\tilde{\rho} = 0$.

Denote $\{\lambda_t(i), \pi_t(i), \mu_t(i)\}$ as the Lagrangian multipliers of constraints (66), (67), and (34), respectively, the firm's first order conditions for $\{I_t(i), K_{t+1}(i)\}$ are given, respectively, by $1 = \varepsilon_t(i)\lambda_t(i) + \pi_t(i) - \mu_t(i)$ and

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{v}_{t+1}. \tag{68}$$

The envelop condition is given by $v_t(i) = R_t + (1-\delta)\lambda_t(i) + \theta Q_t(i)\mu_t(i)$. Substituting this expression into equation (68) gives

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{R_{t+1} + (1-\delta)\lambda_{t+1}(i) + \theta Q_{t+1}(i)\mu_{t+1}(i)\right\}. \tag{69}$$

Hence, the first-order conditions are all the same as those in the benchmark model except here we have $\theta = (1-\delta) Q_t(i)$. Therefore, following the same steps of analysis as in the benchmark model (Section 2.2) by considering different cases for the possible values of the Lagrangian multipliers, it can be easily shown that the Lagrangian multipliers are given by $\mu_t(i) = \max\{q_t(i) - 1, 0\}$, $\pi_t(i) = \max\{1 - q_t(i), 0\}$, where $q_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*}$; and the firm's optimal decision rules for investment and capital accumulation are given by equations (35) and (36), and the firm's value function is given by equation (37). Clearly, since R_t and Q_t are independent of $K_t(i)$, equation (37) implies that the value of a firm is proportional to its capital stock: $V[\varepsilon_t(i), K_t(i)] = v_t(i)K_t(i)$. This confirms our initial conjecture. ■

Appendix III. Proof of Proposition 3

Proof. Denote $i_t \equiv \frac{I_t}{K_t}$, then

$$\frac{\partial \Psi}{\partial i_t} = (1-\delta)\frac{\partial Q}{\partial i_t}\int_{\varepsilon \geq \frac{1}{Q(i_t)}} \varepsilon d\Phi(\varepsilon) + (1-\delta)\phi(\varepsilon_t^*)Q_t^{-2}\frac{\partial Q}{\partial i_t}. \tag{70}$$

Since $Q_t = \frac{1}{\varepsilon_t^*}$, equation (38) implies

$$\frac{\partial i_t}{\partial Q_t} = (1-\delta)\left[1 - \Phi(\varepsilon_t^*)\right] + (1-\delta)\varepsilon_t^*\phi(\varepsilon_t^*). \tag{71}$$

The above two equations together imply

$$\Psi'(i_t) = \varepsilon_t^* \left[\frac{\int_{\varepsilon \geq \varepsilon^*} \frac{\varepsilon}{\varepsilon^*} d\Phi(\varepsilon) + \varepsilon_t^* \phi(\varepsilon_t^*)}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^* \phi(\varepsilon_t^*)} \right] > \varepsilon_t^* > 0, \tag{72}$$

where the inequality holds because $\int_{\varepsilon \geq \varepsilon^*} \frac{\varepsilon}{\varepsilon^*} d\Phi(\varepsilon) > 1 - \Phi(\varepsilon^*)$ and the support of ε is in the positive region of the real line.

Integration by parts and rearranging, the first term in the numerator of $\Psi'(i)$ can be written as $\int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) = \varepsilon_t^*[1 - \Phi(\varepsilon^*)] + \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon$. Thus,

$$\begin{aligned} \Psi^{'}(i_t) &= \frac{\varepsilon_t^*[1 - \Phi(\varepsilon^*)] + \varepsilon_t^{*2} \phi(\varepsilon_t^*) + \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^* \phi(\varepsilon_t^*)} = \varepsilon_t^* + \frac{\int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^* \phi(\varepsilon_t^*)} \\ &\equiv f(\varepsilon^*). \end{aligned} \tag{73}$$

Notice that

$$\begin{aligned} f'(\varepsilon_t^*) &= 1 + \frac{-[1 - \Phi(\varepsilon_t^*)] \{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\} - \varepsilon_t^* \phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{\{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2} \\ &= 1 - \frac{[1 - \Phi(\varepsilon_t^*)]}{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]} - \frac{\varepsilon_t^* \phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{\{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2} \\ &= \frac{\varepsilon_t^* \phi(\varepsilon_t^*)}{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]} - \frac{\varepsilon_t^* \phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{\{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2}. \end{aligned} \tag{74}$$

Clearly, so long as $\phi'(\varepsilon_t^*) \leq 0$, we have

$$f'(\varepsilon_t^*) \geq 0 \tag{75}$$

and

$$\Psi^{''}(i_t) = f'(\varepsilon_t^*) \frac{\partial \varepsilon_t^*}{\partial i_t} \leq 0 \tag{76}$$

since $\frac{\partial \varepsilon_t^*}{\partial i_t} < 0$ by equation (38). Therefore, $\Psi(\cdot)$ is increasing and concave. In addition, it is clear that $\Psi(\cdot)$ depends only on the investment to capital ratio i_t, so it is homogenous of degree zero in $\{I, K\}$. ■

Appendix IV. Proof of Proposition 4

Proof. Consider a representative firm solving the program in equation (28) subject to equation (45), taking $\bar{\imath}_t$ as given. Denoting Q_t as the Lagrangian multiplier for the constraint and imposing the equilibrium condition $\bar{\imath}_t = i_t$, the first order condition for I_t and K_{t+1} are given, respectively, by

$$\theta Q_t^2 \varphi'(i_t) = 1 \tag{77}$$

$$Q_t = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1} + (1-\delta) Q_{t+1} + \theta Q_{t+1}^2 [\varphi(i_t) - \varphi'(i_{t+1}) i_{t+1}] \right\}. \tag{78}$$

Since $\theta Q_{t+1}^2 \varphi'(i_{t+1}) = 1$, equation (78) can be written as

$$Q_t = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1} + (1-\delta) Q_{t+1} + \theta Q_{t+1}^2 \int_{\varepsilon \geq \frac{1}{Q(i_{t+1})}} \varepsilon d\Phi(\varepsilon) - \frac{I_{t+1}}{K_{t+1}} \right\}, \tag{79}$$

which is identical to equation (39) in the microfounded model. ■

Appendix V. Proof of Proposition 5

Proof. The first part of the proposition—the optimal capital tax rate in the representative-agent CAC model without externalities is optimal—is a standard result in the literature. Hence, we only need to prove the second part of the proposition. We add a representative household into the model so the government's objective function is well-defined. We prove the proposition in an environment without aggregate uncertainty. The household's problem is to choose consumption (C_t) and labor supply (N_t) in each period to solve

$$\max \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] \tag{80}$$

subject to $C_t \leq w_t N_t + \Pi_t + T_t$, where Π_t denotes aggregate dividend distributed from firms and $T_t = \int \tau_t [Y_t(i) - w_t N_t(i)] di$ is a lump sum transfer from the government based on capital tax revenues collected from all firm i, where τ_t is the tax rate for capital income. The first-order conditions of the household can be summarized by

$$u'(C_t) w_t = v'(N_t). \tag{81}$$

On the firm side, we can show that, regardless of capital tax, the endogenous credit limit model is always equivalent to a representative-firm model with investment externality. Hence, based on the equivalence, we only need to prove that the optimal capital tax rate is negative in the representative-firm model with investment externality. For simplicity, we consider Pareto distribution for firms' idiosyncratic shocks $\varepsilon_t(i)$ (in the microfounded model) and Cobb-Douglas production function, $Y_t = AK_t^{\alpha} N_t^{1-\alpha}$. Thus, the equivalent CAC function is a Cobb-Douglas form and the problem of a representative firm in the investment externality model is to solve

$$\max \sum_{\tau=0}^{\infty} \beta^{\tau} \frac{\Lambda_{t+\tau}}{\Lambda_t} \{(1-\tau_t)(Y_t - w_t N_t) - I_t\} \tag{82}$$

subject to

$$K_{t+1} = (1-\delta)K_t + \varphi_0 \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^{a} K_t^{1-b} I_t^{b}, \tag{83}$$

where $a = \frac{1}{\eta(\eta+1)}$, $b = \frac{\eta-1}{\eta}$, and $\frac{\bar{I}_t}{\bar{K}_t}$ denotes the average investment rate in the economy that each firm takes as given. The first order conditions for $\{I_t, K_{t+1}\}$ in this model are given by

$$Q_t \varphi_0 b \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^{a} K_t^{1-b} I_t^{b-1} = 1 \tag{84}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{(1-\tau_{t+1})R_{t+1} + (1-\delta)Q_{t+1} + Q_{t+1}\varphi_0(1-b)\left(\frac{\bar{I}_t}{\bar{K}_t}\right)^{a} K_{t+1}^{-b} I_{t+1}^{b}\right\}, \tag{85}$$

where $R_t = \alpha \frac{Y_t}{K_t}$ and $w_t = \frac{(1-\alpha)Y_t}{N_t}$. Imposing the equilibrium conditions, $\frac{\bar{I}_t}{\bar{K}_t} = \frac{I_t}{K_t}$ and $\Lambda_t = u'(C_t)$, and plugging in the values of $\{\varphi_0, a, b\}$, and substituting out $\{R_t, w_t, Q_t\}$, the above two first-order conditions become

$$\frac{I_t}{K_t} = \theta Q_t^{1+\eta} \tag{86}$$

$$\theta^{-\frac{1}{\eta+1}} \left(\frac{I_t}{K_t}\right)^{\frac{1}{\eta+1}} = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[(1-\tau_{t+1})\frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta)\theta^{-\frac{1}{\eta+1}} \left(\frac{I_{t+1}}{K_{t+1}}\right)^{\frac{1}{\eta+1}} + \frac{1}{\eta-1}\frac{I_{t+1}}{K_{t+1}}\right] \tag{87}$$

The law of motion for capital accumulation becomes

$$K_{t+1} = (1-\delta)K_t + \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta-1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}}, \tag{88}$$

and the household resource constraint becomes

$$I_t + C_t = Y_t = AK_t^{\alpha} N_t^{1-\alpha}. \tag{89}$$

Notice that equations (81), (87), (88), (89), and the aggregate production function can uniquely pin down the competitive equilibrium path of $\{C_t, I_t, Y_t, N_t, K_{t+1}\}$ as a function of the tax rate τ_t in the externality model. The optimal tax policy is to design a sequence of tax rates $\{\tau_t\}_{t=0}^{\infty}$ to solve

$$V(K_0) = \max_{\{\tau_t\}} \sum_{t=0}^{\infty} \beta^t [u(C(\tau_t) - v(N(\tau_t)] \tag{90}$$

subject to equations 81, (87), (88), (89), and the aggregate production function.

Instead of directly solving program (90), we first study the "first best allocation" in the externality model, which pertains to the highest possible utility that a social planner can achieve in the model when the investment externality is fully endogenized. Hence, the first best allocation also pertains to the highest possible utility that the government can achieve using tax policies in program (90).

The first best allocation solves

$$V^*(K_0) = \max_{\{C_t, N_t, I_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] \tag{91}$$

subject to

$$K_{t+1} = (1-\delta)K_t + \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta - 1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}} \tag{92}$$

$$C_t + I_t = AK_t^{\alpha} N_t^{1-\alpha} \tag{93}$$

It is obvious that the lifetime utility defined in program (91) is at least as large as that defined in program (90): $V^*(K_0) \geq V(K_0)$, because the former gives the first best allocation. The first-order conditions for $\{I_t, C_t, K_{t+1}\}$ in program (91) are given, respectively, by

$$Q_t \frac{\eta}{\eta+1} \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta-1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}-1} = 1 \tag{94}$$

$$u'(C_t) \frac{(1-\alpha)Y_t}{N_t} = v'(N_t) \tag{95}$$

$$Q_t = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[\frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta) Q_{t+1} + Q_{t+1} \frac{1}{\eta+1} \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta - 1} K_t^{\frac{1}{\eta+1}-1} I_t^{\frac{\eta}{\eta+1}} \right]. \tag{96}$$

The first equation above implies $Q_t = \frac{\eta^2-1}{\eta^2} \theta^{-\frac{1}{\eta+1}} \left(\frac{I_t}{K_t} \right)^{\frac{1}{\eta+1}}$. Using this relationship to substitute out Q, equation (96) becomes

$$\frac{\eta^2-1}{\eta^2} \theta^{-\frac{1}{\eta+1}} \left(\frac{I_t}{K_t} \right)^{\frac{1}{\eta+1}} = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[\frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta) \frac{\eta^2-1}{\eta^2} \theta^{-\frac{1}{\eta+1}} \left(\frac{I_{t+1}}{K_{t+1}} \right)^{\frac{1}{\eta+1}} + \frac{1}{\eta} \frac{I_{t+1}}{K_{t+1}} \right]. \tag{97}$$

Notice that equations (95), (97), (92), (93), and the aggregate production function together uniquely solve for the first best allocation $\{C_t, I_t, Y_t, N_t, K_{t+1}\}$ under program (91). Similarly, equations (81), (87), (88), (89), and the aggregate production function together uniquely solve for the equilibrium path of $\{C(\tau_t), I(\tau_t), Y(\tau_t), N(\tau_t), K(\tau_t)\}$ in a competitive equilibrium with investment externalities. Comparing these two system of equations, except that equation (97) is different from equation (87), all other equilibrium conditions in the first best allocation are identical to those in a competitive equilibrium in terms of mathematical relationship. In particular, equations (95), (92), and (93) are identical to equations (81), (88), and (89), respectively.

Denote the equilibrium path of the first best allocation as $\{C_t^*, I_t^*, Y_t^*, N_t^*, K_{t+1}^*\}$. Compare equation (97) under program (91) with equation (87) in the competitive equilibrium, it is obvious that the government can achieve the first best allocation in program (90) by setting the tax rate such that equation (97) and equation (87) are identical, which implies

$$\begin{aligned} & \frac{\eta^2}{\eta^2-1} \frac{\alpha Y_{t+1}^*}{K_{t+1}^*} + (1-\delta) \theta^{-\frac{1}{\eta+1}} \left(\frac{I_{t+1}^*}{K_{t+1}^*} \right)^{\frac{1}{\eta+1}} + \frac{\eta}{\eta^2-1} \frac{I_{t+1}^*}{K_{t+1}^*} \\ = \; & (1-\tau_{t+1}) \frac{\alpha Y_{t+1}^*}{K_{t+1}^*} + (1-\delta) \theta^{-\frac{1}{\eta+1}} \left(\frac{I_{t+1}^*}{K_{t+1}^*} \right)^{\frac{1}{\eta+1}} + \frac{1}{\eta - 1} \frac{I_{t+1}^*}{K_{t+1}^*}. \end{aligned} \tag{98}$$

Simplification gives

$$\left(\frac{1}{\eta^2-1} + \tau_{t+1}\right) \alpha Y_{t+1}^* = \frac{1}{\eta^2-1} I_{t+1}^*. \tag{99}$$

Since $Q_t^* = \frac{\eta^2-1}{\eta^2} \theta^{-\frac{1}{\eta+1}} \left(\frac{I_t^*}{K_t^*} \right)^{\frac{1}{\eta+1}}$, we have $Q_{t+1}^* \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta-1} K_{t+1}^{\frac{1}{\eta+1}} I_{t+1}^{\frac{\eta}{\eta+1}} = \frac{\eta+1}{\eta} I_{t+1}^*$. So we can rewrite equation (97) by multiplying both sides by K_{t+1}^* as

$$Q_t^* K_{t+1}^* = \beta \frac{u'(C_{t+1}^*)}{u'(C_t^*)} \left[\alpha Y_{t+1}^* - I_{t+1}^* + Q_{t+1}^* K_{t+2}^* \right]. \tag{100}$$

This equation implies that in the steady state we must have $\alpha Y^* > I^*$. Then by equation (99), we must have $\tau < 0$ in the steady state to achieve the first best allocation. ■

Appendix VI. Model Simulation

1) Simulating aggregate variables.

We solve the equilibrium path of the aggregate variables by log-linear approximation around the deterministic steady state. The log-linearized variable is defined as

$$\hat{x}_t \equiv \log(X_t) - \log \bar{X}, \tag{101}$$

where $\bar{X}$ indicates steady-state value. We simulate the aggregate model for $t = 200,000$ periods using the law of motion of aggregate technology in equation (58). Based on the simulated variables, we can use the following transformation to obtain

$$X_t = \bar{X} \exp(\hat{x}_t). \tag{102}$$

In this way, we obtain the sequences of capital K_t, aggregate investment I_t, Tobin's Q_t, and the cutoff $\varepsilon_t^* = \frac{1}{Q_t}$.

2) Generating firm data. To do so we need to simulate the idiosyncratic shocks, $\varepsilon_t(i)$. A random sample with 200,000 observations for $\varepsilon(i)$ in each time period t can be generated using inverse transform sampling. Given a random variable U drawn from the uniform distribution on the unit interval $(0, 1)$, the variate

$$\varepsilon = \frac{1}{U^{\frac{1}{\eta}}} \tag{103}$$

is Pareto-distributed with the distribution function

$$F(\varepsilon) = 1 - \varepsilon^{-\eta}. \tag{104}$$

Given the sequences of aggregate variables (especially the cutoff ε_t^*), we obtain firm-level investment based on the firm's decision rule,

$$I_t(i) = \begin{cases} Q_t \theta K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}. \tag{105}$$

We normalize each firm's initial capital stock to the aggregate steady-state capital $\bar{K}$; namely, $K_0(i) = \bar{K}$. We construct the firm-level capital sequence by the law of motion:

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i)I_t(i). \tag{106}$$

In each time period $t = 0, 1, ..., 200,000$, we track each firm i's capital stock and positive investment level whenever $\varepsilon_t(i) \geq \varepsilon_t^*$.

3) Regression analysis.

We run two regressions. The first is based on aggregate time series:

$$\frac{K_{t+1} - (1-\delta)K_t}{K_t} = \beta_0 + \beta_1 Q_t. \tag{107}$$

The second is based on firm-level data:

$$\frac{K_{t+1}(i) - (1-\delta)K_t(i)}{K_t(i)} = \beta_0 + \beta_1 Q_t. \tag{108}$$

The adjusted R^2 is almost the same if we used logged variables for the aggregate model. For the firm-level data, since $\frac{K_{t+1}(i)-(1-\delta)K_t(i)}{K_t(i)}$ can be zero in some periods, we can not use logged values in the regression.

References

[1] Abel, A., 1979, Investment and the Value of Capital (Garland, New York).

[2] Abel, A., 1983, Optimal Investment Under Uncertainty, American Economic Review 73(1), 228-233.

[3] Abel, A. and J. Eberly, 1994, A unified model of investment under uncertainty, American Economic Review 84, 1369–1384.

[4] Abel, A. and J. Eberly, 1996, Optimal investment with costly reversibility, Review of Economic Studies 63 581–593.

[5] Bachmann, R., Caballero, R., and Engel, E., 2008, Aggregate Implications of Lumpy Investment: New Evidence and a DSGE Model, NBER working paper No. 12336.

[6] Barlevy, G., 2004, The Cost of Business Cycles under Endogenous Growth, American Economic Review 94(4), 964-990.

[7] Baxter, M. and M. Crucini, 1993, Explaining Saving–Investment Correlations, American Economic Review 83(3), 416-436.

[8] Beaudry, P. and F. Portier, 2007, When can changes in expectations cause business cycle fluctuations in neoclassical settings? Journal of Economic Theory 135, 458-477.

[9] Bernanke, B. and M. Gertler, 1989, Agency costs, net worth, and business fluctuations, American Economic Review 79(1), 14-31.

[10] Bloom, N., S. Bond, and J. Reenen, 2007, Uncertainty and investment dynamics, Review of Economic Studies 74, 391-415.

[11] Boldrin, M., L. Christiano, and J. Fisher, 2001, Habit persistence, asset returns, and the business cycle, American Economic Review 91(1), 149-166.

[12] Caballero, R., 1999, Aggregate investment, in J. Taylor and M. Woodford (eds.) Handbook of Macroeconomics (Amsterdam: North-Holland) 813–862.

[13] Caballero, R., E. Engel, and J. Haltiwanger, 1995, Plant-level adjustment and aggregate investment dynamics, Brookings Papers on Economic Activity 2, 1-39.

[14] Carlstrom C. and T. Fuerst, 1997, Agency costs, net Worth, and business fluctuations: A computable general equilibrium analysis, American Economic Review 87(5), 893-910.

[15] Christiano, L., M. Eichenbaum and C. Evans, 2005, Nominal rigidities and the dynamic effects of a shock to monetary policy, Journal of Political Economy 113(1), 1-45.

[16] Cooper R., J. Haltiwanger, and L. Power, 1999, Machine replacement and the business cycle: Lumps and bumps, American Economic Review, 89, 921–946.

[17] Cooper R. and J. Haltiwanger, 2006, On the nature of capital adjustment costs, Review of Economic Studies (2006) 73, 611–633.

[18] Dixit, A., 1992, Investment and hysteresis, Journal of Economic Perspectives 6, 107-132.

[19] Dixit, A and R. Pindyck, Investment under Uncertainty, Princeton University Press, Princeton, N.J.

[20] Doms, M. and T. Dunne, 1998, Capital adjustment patterns in manufacturing plants, Review of Economic Dynamics 1(2), 409-429.

[21] Gomes, J., 2001, Financing Investment, American Economic Review 91, 1263-1285.

[22] Gould, J., 1968, Adjustment costs in the theory of investment of the firm, Review of Economic Studies 35, 47-55.

[23] Gourio, F. and A. Kashyap, 2007, Investment Spikes: New Facts and a General Equilibrium Exploration, Journal of Monetary Economics, 54: Supplement 1, pp. 1-22.

[24] Hayashi, F., 1982, Tobin's Marginal q and Average q: A Neoclassical Interpretation, Econometrica 50(1), 213-224.

[25] Hassett, K.A. and R.G. Hubbard, 1997, Tax policy and investment, in Fiscal Policy, eds. by A. Auerbach, Massachusetts Institute of Technology (also NBER Working Paper No. 5683).

[26] Jermann, U., 1998, Asset pricing in production economies, Journal of Monetary Economics 41, 257-275.

[27] Khan, A. and Thomas, J., 2003, Nonconvex Factor Adjustments in Equilibrium Business Cycle Models: Do Nonlinearities Matter? Journal of Monetary Economics 50, 331-360.

[28] Khan, A. and Thomas, J., 2008, Idiosyncratic Shocks and the Role of Nonconvexities in Plant and Aggregate Investment Dynamics, Econometrica, 76(2), 395-436.

[29] Krusell, P. and A. Smith, 1998, Income and wealth heterogeneity in the macroeconomy, Journal of Political Economy 106(5), 867-896.

[30] Lorenzoni, G. and K. Walentin, Financial Frictions, Investment and Tobin's q, MIT Working Paper.

[31] Lucas, R., Jr. and E. Prescott, 1971, Investment under uncertainty, Econometrica 39, 659-681.

[32] Pindyck, R., 1991, Irreversibility, uncertainty, and investment, Journal of Economic Literature 26, 1110-1148.

[33] Power, L., 1994, Causes and Consequences of Investment Spikes in U.S. Manufacturing Plants, Ph.D. thesis, University of Maryland.

[34] Smets F. and R. Wouters, 2003, An estimated dynamic stochastic general equilibrium model of the euro area, Journal of European Economic Association 1(5), 1123-1175.

[35] Thomas, J., 2002, Is Lumpy Investment Relevant for the Business Cycle? Journal of Political Economy 110, 508-534.

[36] Tobin, J., 1969, A general equilibrium approach to monetary theory, Journal of Money, Credit and Banking 1(1), 15-29.

[37] Veracierto, M.L., 2002, Plant-Level Irreversible Investment and Equilibrium Business Cycles, American Economic Review 92, 181-197.